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MATHÉMATIQUES

DIFFÉRENTES CLASSES DE FONCTIONS SUBHARMONIQUES CONSIDÉRÉES AU POINT DE VUE DE LEUR REPRÉSENTATION ANALYTIQUE

Par I. I. PRIVALOV

(Présenté par I. M. Vinogradow, de l'Académie, le 21.I.1938)

1. Désignons par $u(P)=u(r, \theta_1, \theta_2, \ldots \theta_{p-1})$ une fonction subharmonique à l'intérieur d'une sphère de rayon 1 ayant pour centre l'origine des coordonnées, et située dans l'espace à $p \geqslant 2$ dimensions. Nous considérons trois classes de fonctions subharmoniques, définies par les conditions suivantes:

$$\int u^+(r, \theta_1, \theta_2, \ldots \theta_{p-1})\, d\omega < K, \quad r < 1, \tag{A}$$

$$\int_e u^{+q}(r, \theta_1, \theta_2, \ldots \theta_{p-1})\, d\omega, \quad q \geqslant 1, \tag{B_q}$$

est une fonction uniformément absolument continue de l'ensemble e pour $r < 1$;

$$\int_e |u(r, \theta_1, \theta_2, \ldots \theta_{p-1})|\, d\omega \tag{C}$$

est une fonction uniformément absolument continue de l'ensemble e au voisinage de $r=1$.

En tenant compte des conditions (A), (B_q) et (C), nous désignerons par $d\omega$ un élément de la sphère unitaire, par e tout ensemble mesurable de points de la sphère unitaire.

J'ai démontré [1], que toute fonction subharmonique $u(P)$ de la classe A (donc a fortiori toute fonction des classes B_q et C) a des valeurs limites $u(Q)$ presque partout sur la sphère unitaire quand on s'approche de sa surface en suivant un rayon et ces valeurs limites $u(Q)$ forment une fonction sommable.

Dans le présent article nous démontrons les propositions suivantes:

I. Les critères nécessaires et suffisants pour qu'une fonction $u(P)$ appartienne aux classes A, B_q, C sont respectivement:

a) $\lim\limits_{r \to 1} \int u^+(P)\, d\omega$ est égale à un nombre fini, ou bien, ce qui est le même

$$\lim_{r \to 1} \int |u(P)|\, d\omega \text{ est égale à un nombre fini.} \tag{A}$$

Les limites faisant partie de ces conditions ne coïncident pas en général avec les nombres $\int u^+(Q)\,d\omega$ et $\int |u(Q)|\,d\omega$ respectivement, elles sont au moins égales à ces nombres.

b) $\lim\limits_{r\to 1} \int u^{+q}(P)\,d\omega = \int u^{+q}(Q)\,d\omega$; (B_q)

c) $\lim\limits_{r\to 1} \int |u(P)|\,d\omega = \int |u(Q)|\,d\omega$. (C)

II. Théorème fondamental. Pour qu'une fonction subharmonique $u(P)$ appartienne à l'une des classes A, B_q, C, il faut et il suffit que sa meilleure majorante harmonique appartienne à cette classe. Par suite le problème de la représentation analytique des fonctions subharmoniques des classes A, B_q, C se réduit à l'étude des représentations analytiques des fonctions harmoniques, appartenant aux classes A, B_q, C.

J'ai demontré dans un article précédent (1), qu'une fonction harmonique de la classe A est caractérisée par sa représentation analytique au moyen d'une intégrale de Poisson-Stiltjes de la forme générale; une fonction harmonique de la classe C par sa représentation analytique au moyen d'une intégrale de Poisson. Enfin, dans la présente note on démontre qu'une fonction harmonique de la classe B_q est caractérisée par sa représentation analytique au moyen d'une intégrale de Poisson-Stiltjes d'une forme particulière, c'est le cas où l'on a sous le signe de la différentielle la différence d'une fonction absolument continue d'ensemble * et d'une fonction non-négative d'ensemble dont la dérivée est nulle presque partout sur la sphère unitaire. De plus une fonction de la classe B_q, $q > 1$, peut être caractérisée par l'inégalité

$$\int u^{+q}(P)\,d\omega < K, \quad r < 1.$$

En supposant en particulier $u(P) = \ln|f(z)|$, ou $f(z)$ est une fonction analytique à l'intérieur d'un cercle unitaire, on obtient respectivement les classes A, B_q, C des fonctions analytiques.

2. Désignons par $u(P) = u(r, \theta_1, \theta_2, \ldots \theta_{p-1})$ une fonction logarithmiquement subharmonique à l'intérieur d'une sphère unitaire de l'espace à $p \geqslant 2$ dimensions.

Désignons par H_δ la classe des fonctions logarithmiquement subharmoniques, définies par la condition suivante:

$$\int_e u^\delta(r, \theta_1 \theta_2, \ldots \theta_{p-1})\,d\omega$$

est une fonction uniformément absolument continue de l'ensemble e pour $r < 1$.

Dans cette condition $d\omega$ désigne un élément de la sphère unitaire, e—un ensemble mesurable quelconque de points de la sphère unitaire, $\delta > 0$—un nombre constant.

En vertu d'un théorème que j'ai démontré précédemment (1) toute fonction subharmonique $u(P)$ de la classe H_δ a des valeurs limites $u(Q)$ presque partout sur la sphère unitaire, quand on suit ces rayons, $u^\delta(Q)$ formant une fonction sommable.

---

* La dérivée $u(Q)$ de cette fonction absolument continue est sommable avec $u^{+q}(Q)$.

On déduit maintenant un critère nécessaire et suffisant pour que la fonction $u(P)$ appartienne à la classe H_δ:

$$\lim_{r\to 1}\int u^\delta(P)\,d\omega=\int u^\delta(Q)\,d\omega.$$

Ensuite on déduit une représentation analytique des fonctions de la classe H_δ qui les caractérise

$$u(P)=\exp-\int G(P;T)\,d\mu(T)\cdot\exp\frac{\Gamma\left(\frac{p}{2}\right)}{2\pi^{\frac{p}{2}}}\int\frac{(1-r^2)\ln u(Q)}{(1+r^2-2r\cos\gamma)^{\frac{p}{2}}}\,d\omega\times$$

$$\times\exp-\frac{\Gamma\left(\frac{p}{2}\right)}{2\pi^{\frac{p}{2}}}\int\frac{1-r^2}{(1+r^2-2\cos\gamma)^{\frac{p}{2}}}\,d\psi_2(e),$$

où $\ln u(Q)$ et $u^\delta(Q)$ sont sommables: $\psi_2(e)\geqslant 0$ et admet une dérivée nulle presque partout sur la sphère unitaire; $G(P;T)$ est une fonction de Green pour la sphère unitaire: $\mu(T)$—la distribution des masses auxquelles la fonction $\ln u(P)$ donne lieu.

Enfin, nous trouvons une autre condition nécessaire et suffisante pour que la fonction $u(P)$ appartienne à la classe H_δ.

$$\int u^\delta(P)\,d\omega<K,\qquad r<1.$$

En supposant en particulier $u(P)=|f(z)|$, où $f(z)$ est une fonction analytique à l'intérieur du cercle unitaire, nous obtenons la classe connue H_δ des fonctions analytiques considérée par Riesz.

3. Appelons intégrale du type Cauchy-Stieltjes l'expression

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{x\,d\psi(\theta)}{x-z},\tag{1}$$

où $x=e^{i\theta}$, $z\neq x$, $\psi(\theta)=\psi_1(\theta)+i\psi_2(\theta)$, $\psi_1(\theta)$ et $\psi_2(\theta)$ étant des fonctions quelconques à variation bornée sur le segment $[0,2\pi]$.

L'expression (1) sera nommée intégrale de Cauchy-Stieltjes, si ses valeurs limites obtenues en passant de l'intérieur du cercle C, $|x|=1$, à sa circonference suivant un chemin quelconque non tangent coïncident avec la dérivée $\psi'(\theta)$ presque partout sur la circonférence.

Théorème 1. Les conditions

$$\int_0^{2\pi}x^n\,d\psi(\theta)=0\quad(n=1,2,\ldots)$$

sont nécessaires et suffisantes pour que l'intégrale du type Cauchy-Stieltjes (1) se transforme en une intégrale de Cauchy-Stieltjes.

Théorème 2. Les conditions

$$\int_0^{2\pi}x^n\,d\psi(\theta)=0\quad(n=1,2,\ldots)$$

sont nécessaires et suffisantes pour que l'intégrale correspondante de Poisson-Stieltjes représente une fonction holomorphe à l'intérieur du cercle unitaire.

Théorème 3. Si l'on a:

$$\int_0^{2\pi} x^n \, d\psi(\theta) = 0 \quad (n = 1, 2, \ldots)$$

la fonction $\psi(\theta)$ est absolument continue.

On déduit de ces théorèmes que si une fonction holomorphe $f(z)$ peut être représentee à l'intérieur du cercle unitaire au moyen de l'une des quatre formules:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(x)\,dx}{x - z} \quad \text{(intégrale de Cauchy);}$$

$$f(z) = \frac{1}{2\pi i} \int_0^{2\pi} f(e^{i\theta}) \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \varphi)}\, d\theta \quad \text{(intégrale de Poisson);}$$

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \varphi)}\, d\psi(\theta) \quad \text{(intégrale de Poisson-Stieltjes);}$$

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{x\, d\psi(\theta)}{x - z} \quad \text{(intégrale de Cauchy-Stieltjes),}$$

cette fonction peut être représentée au moyen des trois autres.

Il en résulte, en particulier, que les fonctions analytiques de la classe H_1 sont caractérisées par la possibilité d'être représentées par une intégrale de Cauchy, ou bien ce qui est le même, par les conditions limites [2]:

$$\int_C f(x)\, x^n\, dx = 0 \quad (n = 0, 1, 2, \ldots).$$

En concluant nous établissons encore la formule suivante:

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{x\, d\psi(\theta)}{x - z} \to \frac{2}{2\pi} \int_0^{2\pi} \frac{x\, d\psi(\theta)}{x - x_0} \pm \frac{1}{2}\, \psi'(\theta_0),$$

qui a lieu presque partout sur la circonférence $C, |x| = 1$, lorsque le point z tend par un chemin quelconque non tangent vers le point $x_0 = e^{i\theta_0}$ de la circonférence C (le signe $+$ a lieu dans le cas, où z est à l'interieur de C; le signe moins, lorsque z est en dehors de C).

L'intégrale du second membre de droite est définie comme une intégrale singulière.

Les démonstrations détaillées des résultats de cette note seront exposées en deux articles dans le Bulletin de l'Académie des Sciences de l'URSS.

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MATHEMATIK

ZUR LAPLACE-TRANSFORMATION. X

Von V. S. IGNATOVSKIJ (W. IGNATOWSKY), korrespondierendem Mitglied der Akademie

Wir wollen hier den einfachsten Fall und zwar die Gleichung

$$\frac{\partial^2 u}{c^2 \partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

betrachten, aber mit den Voraussetzungen

$$0 \leqslant x < \infty; \quad 0 \leqslant y < \infty \tag{2}$$

d. h. den Raum $\left(\text{ebenes Problem, da } \frac{\partial}{\partial z} = 0\right)$ zwischen der XZ- und der YZ-Ebene berücksichtigen, welcher innerhalb des Quadranten mit positiven x und y liegt.

Wir untersuchen demnach die Lösung von (1) innerhalb des Winkels von 90°, also mit einer Ecke im Nullpunkt, wobei die Koordinatenachsen mit je einer Ebene zusammenfallen und senkrecht zur Schnittlinie derselben sein werden.

Den Raum innerhalb des Winkels von 90° teilen wir in fünf Teile. (Da wir ein ebenes Problem vor uns haben, so können wir diese Teilung durch Linien in der Zeichenebene bewerkstelligen.)

Der Raum I ist unendlich und wird durch zwei Linien, parallel den Koordinatenachsen, in der Entfernung ct von denselben, gebildet, mit dem Schnittpunkt auf der Halbierenden des Winkels von 90°.

Verlängern wir diese Linien bis zu den Koordinatenachsen, so erhalten wir zwei Räume II und II′, beide unendlich, wobei die längere Seite von II mit der Y-Achse und die längere Seite von II′ mit der X-Achse zusammenfällt.

Es bleibt uns nur noch das Quadrat mit den Seitenlängen ct übrig. Dieses teilen wir in zwei Räumen III und IV durch einen Kreis mit dem Radius ct und dem Mittelpunkt im Koordinatenanfang, wobei IV den letzten enthalten wird.

Wir nehmen nun wegen der Einfachkeit an, was auch für die Anwendungen wichtiger ist, dass

$$\left.\frac{\partial u(x, y, t)}{\partial x}\right|_{x \to 0} = H(y, t) \tag{3}$$

und

$$\left.\frac{\partial u(x, y, t)}{\partial y}\right|_{y \to 0} = H_1(x, t) \tag{4}$$

längs der Y-Achse, resp. der X-Achse gegeben sind. Ferner setzen wir voraus, dass

$$u(x,y,0)=\frac{\partial u(x,y,t)}{\partial t}_{t\to 0}=0 \tag{5}$$

sein werden. Wir bezeichnen nun

$$u(0,y,t)=N(y,t) \tag{6}$$

und

$$u(x,0,t)=N_1(x,t). \tag{7}$$

Ein ähnliches Verfahren, wie in der Arbeit VI (1), ergibt uns die Lösung.

In dem Raume I:

$$u(x,y,t)=0. \tag{8}$$

Im Raume II wird es sein:

$$u(x,y,t)=-\frac{1}{\pi}\int_x^{ct} d\tau_1 \int_{-\sqrt{\tau_1^2-x^2}}^{\sqrt{\tau_1^2-x^2}} \frac{H(y-s,t-\tau)\,ds}{\sqrt{\tau_1^2-x^2-s^2}}; \quad \tau_1=c\tau. \tag{9}$$

Indem wir ähnlich, wie in der Arbeit VI, verfahren, haben wir die Gleichung (1) in den Unterraum bezüglich y transformiert, aber laut (2) die Integration von $y=0$ bis $y=\infty$ geführt. Hierbei sind die Grössen H_1 und N_1 entstanden. In diesem Raum erhalten wir, analog der Arbeit I (2), eine Lösung und der Rückgang in den Oberraum ergibt uns dann die allgemeine Lösung.

Ebensogut können wir in bezug auf x in den Unterraum übergehen. Hierbei ergibt sich dann eine Lösung, die gleich der vorhergehenden sein muss. Dabei sind aber y, H, N mit x, H_1 und N_1 vertauscht. (Man muss nicht vergessen, dass hierbei der Raum II in den Raum II′ übergeht.)

Durch Gleichsetzung dieser Lösungen erhalten wir gewisse Beziehungen. Indem wir diese Vertauschung anwenden, erhalten wir z. B. für den Raum II′:

$$u(x,y,t)=-\frac{1}{\pi}\int_y^{ct} d\tau_1 \int_{-\sqrt{\tau_1^2-y^2}}^{\sqrt{\tau_1^2-y^2}} \frac{H_1(x-s,t-\tau)\,ds}{\sqrt{\tau_1^2-y^2-s^2}}=$$

$$=-\frac{1}{2\pi}\frac{\partial}{\partial y}\int_y^{ct} d\tau_1 \int_{-\sqrt{\tau_1^2-y^2}}^{\sqrt{\tau_1^2-y^2}} \frac{N_1(x-s,t-\tau)\,ds}{\sqrt{\tau_1^2-y^2-s^2}}-$$

$$-\frac{1}{2\pi}\int_y^{ct} d\tau_1 \int_{-\sqrt{\tau_1^2-y^2}}^{\sqrt{\tau_1^2-y^2}} \frac{H_1(x-s,t-\tau)\,ds}{\sqrt{\tau_1^2-y^2-s^2}}. \tag{10}$$

Wegen der Kompliziertheit der Formeln werden wir im folgenden nur die fertigen Lösungen hinschreiben, wobei man sich die Vertauschung hinzudenkt.

Für den Raum III ergibt sich:

$$u(x,y,t) = -\frac{1}{\pi}\int_x^{ct} d\tau_1 \int_{-\sqrt{\tau_1^2-x^2}}^{\sqrt{\tau_1^2-x^2}} \frac{H(y-s,\,t-\tau)\,ds}{\sqrt{\tau_1^2-x^2-s^2}} -$$

$$-\frac{1}{2\pi}\int_y^{ct} d\tau_1 \int_{-\sqrt{\tau_1^2-y^2}}^{\sqrt{\tau_1^2-y^2}} \frac{H_1(x-s,\,t-\tau)\,ds}{\sqrt{\tau_1^2-y^2-s^2}} -$$

$$-\frac{1}{2\pi}\frac{\partial}{\partial y}\int_y^{ct} d\tau_1 \int_{-\sqrt{\tau_1^2-y^2}}^{\sqrt{\tau_1^2-y^2}} \frac{N_1(x-s,\,t-\tau)\,ds}{\sqrt{\tau_1^2-y^2-s^2}}; \quad \tau_1 = c\tau. \tag{11}$$

Für den Raum IV, endlich, erhalten wir:

$$u(x,y,t) = -\frac{1}{\pi}\int_{\sqrt{y^2+x^2}}^{ct} d\tau_1 \int_{-\sqrt{\tau_1^2-x^2}}^{y} \frac{H(y-s,\,t-\tau)\,ds}{\sqrt{\tau_1^2-x^2-s^2}} -$$

$$-\frac{1}{\pi}\int_x^{\sqrt{y^2+x^2}} d\tau_1 \int_{-\sqrt{\tau_1^2-x^2}}^{\sqrt{\tau_1^2-x^2}} \frac{H(y-s,\,t-\tau)\,ds}{\sqrt{\tau_1^2-x^2-s^2}} -$$

$$-\frac{1}{2\pi}\int_{\sqrt{y^2+x^2}}^{ct} d\tau_1 \int_{x}^{\sqrt{\tau_1^2-y^2}} \frac{H_1(s-x,\,t-\tau)\,ds}{\sqrt{\tau_1^2-y^2-s^2}} -$$

$$-\frac{1}{2\pi}\frac{\partial}{\partial y}\int_{\sqrt{y^2+x^2}}^{ct} d\tau_1 \int_{x}^{\sqrt{\tau_1^2-y^2}} \frac{N_1(s-x,\,t-\tau)\,ds}{\sqrt{\tau_1^2-y^2-s^2}} -$$

$$-\frac{1}{2\pi}\int_y^{\sqrt{y^2+x^2}} d\tau_1 \int_{-\sqrt{\tau_1^2-y^2}}^{\sqrt{\tau_1^2-y^2}} \frac{H_1(x-s,\,t-\tau)\,ds}{\sqrt{\tau_1^2-y^2-s^2}} -$$

$$-\frac{1}{2\pi}\frac{\partial}{\partial y}\int_y^{\sqrt{y^2+x^2}} d\tau_1 \int_{-\sqrt{\tau_1^2-y^2}}^{\sqrt{\tau_1^2-y^2}} \frac{N_1(x-s,\,t-\tau)\,ds}{\sqrt{\tau_1^2-y^2-s^2}} -$$

$$-\frac{1}{2\pi}\int_{\sqrt{y^2+x^2}}^{ct} d\tau_1 \int_{-\sqrt{\tau_1^2-y^2}}^{x} \frac{H_1(x-s,\,t-\tau)\,ds}{\sqrt{\tau_1^2-y^2-s^2}} -$$

$$-\frac{1}{2\pi}\frac{\partial}{\partial y}\int_{\sqrt{y^2+x^2}}^{ct} d\tau_1 \int_{-\sqrt{\tau_1^2-y^2}}^{x} \frac{N_1(x-s,\,t-\tau)\,ds}{\sqrt{\tau_1^2-y^2-s^2}}; \quad \tau_1 = c\tau. \tag{12}$$

In (8)—(12) muss man sich N resp. N_1 (ausgedrückt durch H resp. H_1) eingesetzt denken. Die entsprechenden Formeln lauten:

$$N(y,t) = -\frac{1}{\pi}\int_0^{ct} d\tau_1 \int_{-\tau_1}^{\tau_1} \frac{H(y-s, t-\tau)\,ds}{\sqrt{\tau_1^2 - s^2}}\,; \quad \underline{\underline{y \geqslant ct}} \tag{13}$$

und

$$N(y,t) = -\frac{2}{\pi}\int_y^{ct} d\tau_1 \int_0^{\sqrt{\tau_1^2 - y^2}} \frac{H_1(s, t-\tau)\,ds}{\sqrt{\tau_1^2 - y^2 - s^2}} -$$

$$-\frac{1}{\pi}\int_y^{ct} d\tau_1 \int_y^{\tau_1} \frac{H(s-y, t-\tau)\,ds}{\sqrt{\tau_1^2 - s^2}} - \frac{1}{\pi}\int_0^{y} d\tau_1 \int_{-\tau_1}^{\tau_1} \frac{H(y-s, t-\tau)\,ds}{\sqrt{\tau_1^2 - s^2}} -$$

$$-\frac{1}{\pi}\int_y^{ct} d\tau_1 \int_{-\tau_1}^{y} \frac{H(y-s, t-\tau)\,ds}{\sqrt{\tau_1^2 - s^2}}\,; \quad \underline{\underline{y \leqslant ct}};\ \tau_1 = c\tau. \tag{14}$$

Die Werte von H_1 erhalten wir durch obige Vertauschung.

Die Gleichungen (8)—(12) genügen der Gleichung (2), gehen stetig ineinander über und ergeben auf der X-Achse, resp. der Y-Achse, die gegebenen Werte von H und H_1, wie es auch sein muss.. Hierbei müssen wir beachten, dass H und H_1 vollständig voneinander unabhängig sind, wie dies auch klar ist.

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MATHEMATICS

EXISTENCE THEOREMS FOR THE QUASI-DIFFERENTIAL EQUATION OF THE *n*th ORDER

By D. SHIN

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Let $P_{k,\nu}(x), \nu \leqslant k,\ k=0,1,\ldots,n;\ \nu=0,1,\ldots,n$, be complex-valued measurable after Lebesgue functions defined on a finite closed interval (a,b). We introduce the following symbols for an arbitrary complex-valued function $u(x)$:

$$u^{[0]}=P_{0,0}u,\quad u^{[1]}=iP_{1,1}\frac{d}{dx}u^{[0]}+P_{1,0}u^{[0]},$$

$$u^{[2]}=iP_{2,2}\frac{d}{dx}u^{[1]}+P_{2,1}u^{[1]}+P_{2,0}u^{[0]},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$u^{[n]}=iP_{n,n}\frac{d}{dx}u^{[n-1]}+P_{n,n-1}u^{[n-1]}+$$

$$+P_{n,n-2}u^{[n-2]}+\ldots+P_{n,1}u^{[1]}+P_{n,0}u^{[0]}.$$

We assert that if the functions $\left|\frac{1}{P_{k,k}}\right|^2,\ k=0,1,\ldots n,\ |P_{k,\nu}|^2, \nu<k$, and $|f(x)|^2$ are integrable after Lebesgue over (a,b), then there exists a unique function $u(x)$ with the properties:

1) $u^{[k]}(x),\ k=0,1,\ldots n-1$, are equal to the absolutely continuous functions almost everywhere on (a,b),

2) $u^{[n]}(x)=f(x)$ almost everywhere on (a,b), and

3) $u^{[k]}(c)=C_k,\ k=0,1,\ldots n-1$, where $a\leqslant c\leqslant b$, C_k — are given arbitrary complex-valued constants.

We call the function $u(x)$ with the properties 1)—3) a solution of the equation $u^{[n]}(x)=f(x)$ satisfying the initial conditions 3).

Suppose that such a function $u(x)$ exists. Then

$$u^{[n-1]}(x)=C_{n-1}+\int_c^x\frac{1}{iP_{n,n}(\xi)}f(\xi)\,d\xi-\sum_{\nu=0}^{n-1}\int_c^x\frac{P_{n,\nu}(\xi)}{iP_{n,n}(\xi)}u^{[\nu]}(\xi)\,d\xi,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$u^{[k]}(x)=\sum_{j=k}^{n-1}C_j\int_c^x\frac{d\xi_1}{iP_{k+1,k+1}(\xi_1)}\int_c^{\xi_1}\cdots\int_c^{\xi_{j-k-1}}\frac{d\xi}{iP_{j,j}(\xi)}+$$

$$+\int_c^x\frac{d\xi_1}{iP_{k+1,k+1}(\xi_1)}\int_c^{\xi_1}\cdots\int_c^{\xi_{n-k-1}}\frac{1}{iP_{n,n}(\xi)}f(\xi)\,d\xi-$$

$$-\sum_{j=k+1}^{n}\sum_{\nu=0}^{j-1}\int_c^x\frac{d\xi_1}{iP_{k+1,k+1}(\xi_1)}\int_c^{\xi_1}\cdots\int_c^{\xi_{j-k-1}}\frac{P_{j,\nu}(\xi)}{P_{j,j}(\xi)}u^{[\nu]}(\xi)\,d\xi,\quad k=0,1,\ldots n-1.$$

We consider these equalities as a system of integral equations with respect to the functions $u^{[k]}(x), k=0,1,\dots n-1$. If the solutions $u^{[k]}(x)$, $k=0,1,\dots n-1$, of these integral equations exist, then the function $u(x)=\frac{u^{[0]}(x)}{P_{0,0}(x)}$ satisfies the conditions 1) — 3).

Let

$$\varphi_k(x)=\sum_{j=k}^{n-1} C_j \int_c^x \frac{d\xi_1}{iP_{k+1,k+1}(\xi_1)} \int_c^{\xi_1} \cdots \int_c^{\xi_{j-k-1}} \frac{d\xi}{iP_{j,j}(\xi)} +$$

$$+\int_c^x \frac{d\xi_1}{iP_{k+1,k+1}(\xi_1)} \int_c^{\xi_1} \cdots \int_c^{\xi_{n-k-1}} \frac{1}{iP_{n,n}(\xi)} f(\xi)\, d\xi,$$

$$\Gamma_{k,\nu}(x,\xi)=\sum_{j=k+1}^{n} \frac{P_{j,\nu}(\xi)}{iP_{j,j}(\xi)} \int_\xi^x \frac{d\xi_1}{iP_{k+1,k+1}(\xi_1)} \int_\xi^{\xi_1} \cdots \int_\xi^{\xi_{j-k-1}} \frac{d\xi_{j-k-1}}{iP_{j-1,j-1}(\xi_{j-k-1})} =$$

$$=\sum_{j=k+1}^{n} \frac{P_{j,\nu}(\xi)}{iP_{j,j}(\xi)} g_{k,j}(x,\xi), \quad \nu \leqslant k.$$

Thus, our integral equations can be written in the form

$$u^{[k]}(x)=\varphi_k(x)-\sum_{\nu=0}^{n-1}\int_c^x \Gamma_{k,\nu}(x,\xi)\, u^{[\nu]}(\xi)\, d\xi, \qquad k=0,1,\dots n-1.$$

The functions $g_{k,j}(x,\xi)$ are continuous on the finite closed interval $a\leqslant x,\xi\leqslant b$. Hence there exists a constant M such that $|g_{k,j}(x,\xi)|\leqslant M$ for all $k=0,1,\dots n-1$; $j=1,2,\dots n$, and $a\leqslant x,\xi\leqslant b$. On the other hand, the functions $\left|\frac{P_{j,\nu}(\xi)}{P_{j,j}(\xi)}\right|$, $\nu<j$, are integrable after Lebesgue over (a,b). Hence, for an arbitrary positive constant $\rho<1$ there exists a number $\delta=\delta(\rho)$ such that $\sum_{j=1}^{n}\left|\int_\Delta \left|\frac{P_{j,\nu}(\xi)}{P_{j,j}(\xi)}\right| d\xi\right|<\frac{\rho}{nM}$ for all $\nu=0,1,\dots n-1$, provided the length of Δ is less than 2δ. Now we have

$$\sum_{\nu=0}^{n-1}\left|\int_\Delta |\Gamma_{k,\nu}(x,\xi)|\, d\xi\right|<\rho.$$

Let

$$u_0^{[k]}(x)=0, \quad u_{m+1}^{[k]}(x)=\varphi_k(x)-\sum_{\nu=0}^{n-1}\int_c^x \Gamma_{k,\nu}(x,\xi)\, u_m^{[\nu]}(\xi)\, d\xi,$$

$$h_0^{[k]}(x)=0, \quad h_{m+1}^{[k]}(x)=u_{m+1}^{[k]}(x)-u_m^{[k]}(x)=-\sum_{\nu=0}^{n-1}\Gamma_{k,\nu}(x,\xi)\, h_m^{[\nu]}(\xi)\, d\xi.$$

Then the series $\sum_{m=0}^{\infty} h_m^{[k]}(x)$, $k=0,1,\dots n-1$, and, therefore, the sequences of functions $\left\{u_m^{[k]}(x)\right\}$, $k=0,1,\dots n-1$, are uniformly convergent on $(c-\delta, c+\delta)$. It is easy to see that the limit functions $u^{[k]}(x)$, $k=0,1,\dots n-1$, satisfy the integral equations and they are unique on $(c-\delta, c+\delta)$. Now we divide (a,b) into a finite number of intervals $\Delta_1,\Delta_2,\dots\Delta_N$ of

the length less than 2δ overlapping one another. Let the point $x=c$ be contained in Δ_j. We form the solutions $u^{[k]}(x)$, $k=0,1,\dots n-1$, beginning with Δ_j in such a way that they coincide in a point common to two neighbouring intervals. Then they coincide in the whole common part of these two intervals. Thus we have the unique solutions on the whole interval (a,b).

We call the differential expressions:

$$v^{\{0\}}=\overline{P}_{n,n}v,\quad v^{\{1\}}=i\overline{P}_{n-1,n-1}\frac{d}{dx}v^{\{0\}}+\left(\frac{\overline{P_{n,n-1}P_{n-1,n-1}}}{P_{n,n}}\right)v^{\{0\}},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$v^{\{n\}}=i\overline{P}_{0,0}\frac{d}{dx}v^{\{n-1\}}+\sum_{j=0}^{n-1}\left(\frac{\overline{P_{n-j,0}P_{0,0}}}{P_{n-j,n-j}}\right)v^{\{j\}}$$

adjoint with respect to $u^{[k]}(x)$, $k=0,1,\dots n$. It is clear that if the function $|g(x)|^2$ is integrable after Lebesgue over (a,b), then there exists a unique solution of the equation $v^{\{n\}}(x)=g(x)$ with the corresponding properties of 1)—3).

Let $u_1,u_2,\dots u_n$ be solutions of the homogeneous equation $u^{[n]}=0$. Just as in the classical case a necessary and sufficient condition that these functions are linearly independent is that the determinant:

$$[w]_x=\begin{vmatrix} u_1^{[0]} & u_2^{[0]} & \dots & u_n^{[0]} \\ u_1^{[1]} & u_2^{[1]} & \dots & u_n^{[1]} \\ \dots & \dots & \dots & \dots \\ u_1^{[n-1]} & u_2^{[n-1]} & \dots & u_n^{[n-1]} \end{vmatrix}=[w]_c e^{i\int_c^x \sum_{k=1}^{n}\frac{P_{k,k-1}(\xi)}{P_{k,k}(\xi)}d\xi}$$

is different from zero. The functions $v_k(x)=\left(\overline{\frac{1}{P_{n,n}}\frac{\partial\ln[w]_x}{i\partial u_k^{[n-1]}}}\right)$, $k=1,2\dots n$, are solutions of the adjoint homogeneous equation $v^{\{n\}}(x)=0$, and we have

$$v_k^{\{\nu\}}=(-1)^{\nu}\left(\overline{\frac{\partial\ln[w]_x}{i\partial u_k^{[n-\nu-1]}}}\right),\quad \nu=0,1\dots n-1.$$

If the functions u_k are linearly independent, then the functions v_k also must be linearly independent, since we have $[w]_x\{w\}_x=1$, where

$$\{w\}_x=\begin{vmatrix} v_1^{\{0\}} & v_1^{\{1\}} & \dots & v_1^{\{n-1\}} \\ v_2^{\{0\}} & v_2^{\{1\}} & \dots & v_2^{\{n-1\}} \\ \dots & \dots & \dots & \dots \\ v_n^{\{0\}} & v_n^{\{1\}} & \dots & v_n^{\{n-1\}} \end{vmatrix}=\{w\}_c e^{\overline{i\int_c^x \sum_{k=1}^{n}\frac{P_{k,k-1}(\xi)}{P_{k,k}(\xi)}d\xi}}.$$

We note that the functions $u(x)$ and $v(x)$ are solutions of the equations $u^{[n]}(x)=f(x)$, $v^{\{n\}}(x)=g(x)$ respectively, if and only if they can be expressed in the form

$$u(x)=\sum_{k=1}^{n}C_k u_k(x)+\int_c^x\sum_{k=1}^{n}u_k(x)\overline{v_k(\xi)}f(\xi)\,d\xi,$$

$$v(x)=\sum_{k=1}^{n}C_k v_k(x)+\int_c^x\sum_{k=1}^{n}v_k(x)\overline{u_k(\xi)}g(\xi)\,d\xi,$$

where C_k are arbitrary constants.

The bilinear form of the adjoint differential expressions $u^{[k]}$, $v\{k\}$, $k=0,1,\dots n-1$, can be written in the form:

$$\int_a^b (\bar{v}u^{[n-1]} - u\,\overline{v\{n-1\}})\,dx = i\sum_{k=1}^{n} u^{[n-k]}v\{k-1\}\Big|_{x=a}^{x=b}.$$

We call the differential expression $u^{[n]}$ self-adjoint, if $u^{[k]} \equiv u\{k\}$ for all $k=0,1,\dots n$, in other words, if $\overline{P}_{n-k,\,n-k} = P_{k,k}$, $k=0,1,\dots\left[\frac{n}{2}\right]$, where $\left[\frac{n}{2}\right]$ is the integer part of $\frac{n}{2}$, and $\overline{\left(\frac{P_{n-\nu,\,n-k}P_{n-k,\,n-k}}{P_{n-\nu,\,n-\nu}}\right)} = P_{k,\nu}$, $\nu < k$, $k=1,2,\dots n$; $\nu=0,1,\dots n-1$.

It is easy to see that the self-adjoint differential expressions

$$P_0\frac{d}{dx}P_1\frac{d}{dx}\dots P_{n-1}\frac{d}{dx}P_n\frac{d}{dx}P_{n-1}\frac{d}{dx}\dots P_1\frac{d}{dx}P_0u,$$

$$P_0\frac{d}{dx}P_1\frac{d}{dx}\dots P_{n-1}\frac{d}{dx}P_n\frac{d}{dx}P_n\frac{d}{dx}P_{n-1}\frac{d}{dx}\dots P_1\frac{d}{dx}P_0u,$$

$$\sum_{k=0}^{n}\frac{d^{n-k}}{dx^{n-k}}\left(P_k\frac{d^{n-k}u}{dx^{n-k}}\right),\quad \sum_{k=0}^{n}\left[\frac{d^{n+1-k}}{dx^{n+1-k}}\left(P_k\frac{d^{n-k}u}{dx^{n-k}}\right)+\frac{d^{n-k}}{dx^{n-k}}\left(P_k\frac{d^{n+1-k}u}{dx^{n+1-k}}\right)\right],$$

are particular cases.

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MATHEMATICS

ON THE SOLUTIONS OF THE SELF-ADJOINT EQUATION DIFFERENTIAL $u^{[n]}=lu,\ I(l)\neq 0$ in $L_2(0,\ \infty)$

By D. SHIN

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We call that the function $f(x)$ is in $L_2(a, b)$, if it is measurable after Lebesgue on (a, b) and $|f(x)|^2$ is integrable after Lebesgue over (a, b). Let $\frac{1}{P_{k,k}(x)}$, $k=0, 1, \dots n$, $P_{k,\nu}(x)$, $\nu<k$, be complex functions in $L_2(0, b)$, where b is an arbitrary positive number. Then there exist n linearly independent solutions in $L_2(0, b)$ of the equation $u^{[n]}(x)=0$, where

$$u^{[k]}=iP_{k,k}\frac{d}{dx}u^{[k-1]}+\sum_{j=0}^{k-1}P_{k,j}\,u^{[j]},\qquad k=1, 2, \dots n,$$

$$u^{[0]}=P_{0,0}\,u.$$

If the differential expressions $u^{[k]}$, $k=0, 1, \dots n$, are self-adjoint, that is, the functions $P_{k,\nu}$ satisfy the equations:

$$\overline{P_{n-k,n-k}}=P_{k,k},\ k=0, 1, \dots\left[\frac{n}{2}\right],\ \left(\overline{\frac{P_{n-\nu,n-k}\,P_{n-k,n-k}}{P_{n-\nu,n-\nu}}}\right)=$$
$$=P_{k,\nu},\ \nu<k,\ k=1, 2, \dots n;\ \nu=0, 1, \dots n-1,$$

where $\left[\frac{n}{2}\right]$ is the integer part of $\frac{n}{2}$, then the bilinear form can be written as

$$\int_0^b(\bar{v}u^{[n]}-u\overline{v^{[n]}})\,dx=[u, v]_b-[u, v]_0,$$

where

$$[u, v]_x=i\sum_{k=1}^{n}u^{[n-k]}(x)\,\overline{v^{\{k-1\}}(x)},$$

and if the functions u, v are the solutions of the equations $u^{[n]}=lu$, $v^{[n]}=\bar{l}v$, $I(\bar{l})\neq 0$, respectively, then

$$\int_0^b|u|^2dx=\frac{1}{2iI(l)}([u, u]_b-[u, u]_0),\qquad(1)$$

$$\int_0^b|v|^2dx=-\frac{1}{2iI(l)}([v, v]_b-[v, v]_0).$$

Let $u_1, u_2 \dots u_n$ be linearly independent solutions of the equation $u^{[n]} = lu$, $I(l) \neq 0$, such that

$$[u_k, u_{n+1-\nu}]_0 = \delta_{k,\nu} = \begin{cases} i & \text{for } k = \nu, \\ 0 & \text{for } k \neq \nu, \end{cases} \quad k, \nu = 1, 2, \dots n.$$

We assert that the functions $U_k = u_k + \sum_{\nu=1}^{n-\left[\frac{n}{2}\right]} l_{k,\nu} u_{n+1-\nu}$, $k = 1, 2, \dots \left[\frac{n}{2}\right]$, where the complex-valued constants $l_{k,\nu}$ satisfy the equations:

$$\lim_{b \to \infty} [U_k, U_k]_b = 0, \quad k = 1, 2, \dots \left[\frac{n}{2}\right],$$

are in $L_2(0, \infty)$.

The equation $[U_k, U_k]_b = 0$ can be written in the form:

$$[u_k, u_k]_b + \sum_{\nu=1}^{n-\left[\frac{n}{2}\right]} l_{k,\nu} [u_{n+1-\nu}, u_k]_b + \sum_{\nu=1}^{n-\left[\frac{n}{2}\right]} \bar{l}_{k,\nu} [u_k, u_{n+1-\nu}]_b +$$

$$+ \sum_{\nu=1}^{n-\left[\frac{n}{2}\right]} \sum_{j=1}^{n-\left[\frac{n}{2}\right]} l_{k,\nu} \bar{l}_{k,j} [u_{n+1-\nu}, u_{n+1-j}]_b = 0. \tag{2}$$

If we fix the index k and consider $R(l_{k,\nu})\, I(l_{k,\nu})$, $\nu = 1, 2, \dots n - \left[\frac{n}{2}\right]$, as coordinates of Descartes, then the equation (2) means a real closed surface F_b^k of the second order which is placed entirely in a finite part of the space. In fact, it follows from the conditions $[u_k, u_\nu]_x = -\overline{[u_\nu, u_k]}$, $k, \nu = 1, 2, \dots n$, that all the coefficients of the equation (2) are imaginary, and it follows from the equation (1) that the quantities $i[u_{n+1-\nu}, u_{n+1-\nu}]_b$, $\nu = 1, 2, \dots n - \left[\frac{n}{2}\right]$, are of the same sign. On the other hand,

$$|[u_k, u_\nu]_b|^2 < |[u_k, u_k]_b| \cdot |[u_\nu, u_\nu]_b|, \quad k, \nu = 1, 2, \dots n.$$

Hence the projection of the surface F_b^k on each coordinate plane is a circle or an ellipse. It is easy to show that the radius of the circle on the coordinate plane $[R(l_{k,k}),\ I(l_{k,k})]$ is real and equal to

$$r_b^k = \frac{1}{|[u_{n+1-k}, u_{n+1-k}]_b|}.$$

If this circle does not degenerate into a point when $b \to \infty$, then the function u_{n+1-k} is in $L_2(0, \infty)$.

Since $[U_k, U_k]_0 = 2iR(l_{k,k})$, we have:

$$\int_0^b |U_k|^2\, dx = -\frac{R(l_{k,k})}{I(l)},$$

for all points $l_{k,\nu}$ on F_b^k. It is evident that the whole imaginary axis $I(l_{k,k})$ is outside F_b^k, and for all points of this axis we have

$$\int_0^b |U_k|^2\, dx > -\frac{R(l_{k,k})}{I(l)}. \tag{3}$$

Since $\int_0^b |U_k|^2\,dx$ is a continuous function of $l_{k,\nu}$, the inequality (3) holds for all the points outside F_b^k. Hence we have:

$$\int_0^b |U_k|^2\,dx < -\frac{R(l_{k,k})}{I(l)}$$

for all the points $l_{k,\nu}$ inside F_b^k.

Let $b' < b''$. The surfaces $F_{b'}^k$ and $F_{b''}^k$ have no common point, since otherwise there were

$$\int_{b'}^{b''} |U_k|^2\,dx = 0,\ U_k(x) = 0,\ b' \leqslant x \leqslant b''.$$

On the other hand, if the point $l_{k,\nu}$ is on $F_{b''}^k$, then

$$\int_0^{b'} |U_k|^2\,dx < \int_0^{b''} |U_k|^2\,dx = -\frac{R(l_{k,k})}{I(l)},$$

that is, the surface $F_{b''}^k$ is entirely in $F_{b'}^k$. Denote by F^k the limit of F_b^k, $b \to \infty$. Then we have:

$$\int_0^{\infty} |U_k|^2\,dx \leqslant -\frac{R(l_{k,k})}{I(l)}$$

for all the points $l_{k,\nu}$ in or on F^k. Hence, the functions U_k, $k = 1, 2, \ldots \left[\frac{n}{2}\right]$, are in $L_2(0, \infty)$.

Suppose that one of the functions $u_{n+1-\nu}$, $\nu = 1, 2, \ldots \left[\frac{n}{2}\right]$, for example, u_{n+1-k} is in $L_2(0, \infty)$. Then the function $U_k - l_{k,k} u_{n+1-k} = u_k + \sum_{\nu=1\,(\neq k)}^{n-\left[\frac{n}{2}\right]} l_{k,\nu} u_{n+1-\nu}$ is also in $L_2(0, \infty)$. If we put $u_{n+1-j} = u_{n+1-j}^*$, $j \neq k$, $j = 1, 2, \ldots n - \left[\frac{n}{2}\right]$, $u_{n+1-k} + u_{n+1-j} = u_{n+1-k}^*$, the function $U_k^* = u_k + \sum_{\nu=1}^{n-\left[\frac{n}{2}\right]} l_{k,\nu}^* u_{n+1-\nu}^*$, where $l_{k,\nu}^*$ satisfy the equation $\lim_{b\to\infty} [U_k^*, U_k^*]_b = 0$, is in $L_2(0, \infty)$. It is easy to show that $l_{k,\nu}^* = l_{k,\nu}$ for all $\nu \neq k$. Hence the function $l_{k,k}^*(u_{n+1-k} + u_{n+1-j})$ is in $L_2(0, \infty)$; since $l_{k,k}^* \neq 0$, the function u_{n+1-j} is also in $L_2(0, \infty)$, where j is an arbitrary number of $1, 2, \ldots n - \left[\frac{n}{2}\right]$.

Hence, the number p of the solutions of the equation $u^{[n]} = lu$, $I(l) \neq 0$, which are in $L_2(0, \infty)$, is equal either to $\left[\frac{n}{2}\right]$ or $n - \left[\frac{n}{2}\right]$ or n.

In the like manner, the number q of the solutions of the equation $v^{[n]} = \bar{l}v$, $I(l) \neq 0$ which are in $L_2(0, \infty)$, is equal either to $\left[\frac{n}{2}\right]$ or $n - \left[\frac{n}{2}\right]$ or n.

In the case of real differential expressions, that is, $\overline{u^{[n]}} \equiv \bar{u}^{[n]}$, the numbers p and q are equal.

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ON THE QUASI-DIFFERENTIAL TRANSFORMATIONS IN HILBERT SPACE

By D. SHIN

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Let $L_2(a, b)$ be the class of all complex-valued measurable after Lebesgue functions $f(x)$, $a \leqslant x \leqslant b$, for which the Lebesgue integral $\int_a^b |f(x)|^2 dx$ exists. $L_2(a, b)$ forms a Hilbert space, where the scalar product of two elements f and g is defined by the equation

$$(f\,g) = \int_a^b f(x)\,\bar{g}(x)\,dx.$$

Suppose that the complex-valued functions $P_{k,\nu}(x)$, $\nu \leqslant k$, $k = 0, 1, \ldots n$, $\nu = 0, 1, \ldots n$, are measurable after Lebesgue on the finite or infinite interval (a, b), and the functions $\frac{1}{P_{k,k}(x)}$, $k = 0, 1, \ldots n$, $P_{k,\nu}(x)$, $\nu < k$, $k = 1, 2, \ldots n$; $\nu = 0, 1, \ldots n-1$, belong to $L_2(a', b')$, where (a', b') is an arbitrary finite closed interval interior to (a, b).

Introduce the following designations:

1) D_F is the set of all functions $f(x)$ in $L_2(a, b)$ such that

$$f^{[0]} = P_{0,0} f,$$

$$f^{[k]} = i\,P_{k,k}\frac{d}{dx} f^{[k-1]} + \sum_{j=0}^{k-1} P_{k,j}\, f^{[j]}, \quad k = 1, 2, \ldots n-1,$$

are equal to absolutely continuous functions almost everywhere on (a, b), and

$$f^{[n]} = i\,P_{n,n}\frac{d}{dx} f^{[n-1]} + \sum_{j=0}^{n-1} P_{n,j}\, f^{[j]}.$$

belongs to $L_2(a, b)$.

2) F is the transformation defined in the domain D_F which transforms $f(x)$ into $f^{[n]}(x)$: $Ff = f^{[n]}$.

3) D_G is the set of all functions $g(x)$ in $L_2(a,b)$ such that

$$g^{\{0\}} = \overline{P_{n,n}}\, g,$$

$$g^{\{k\}} = i\overline{P_{n-k,\,n-k}}\frac{a}{dx}g^{\{k-1\}} +$$

$$+\sum_{j=0}^{k-1}\overline{\left(\frac{P_{n-j,\,n-k}P_{n-k,\,n-k}}{P_{n-j,\,n-j}}\right)}g^{\{j\}},\quad k=1,2,\ldots n-1,$$

are equal to absolutely continuous functions almost everywhere on (a,b), and

$$g^{\{n\}} = i\overline{P}_{0,0}\frac{d}{dx}g^{\{n-1\}} + \sum_{j=0}^{n-1}\overline{\left(\frac{P_{n-j,\,0}P_{0,\,0}}{P_{n-j,\,n-j}}\right)}g^{\{j\}}$$

belongs to $L_2(a,b)$.

4) G is the transformation defined in the domain D_G which transforms $g(x)$ into $g^{\{n\}}(x)$: $Gg = g^{\{n\}}$.

5) $$[fg] = (Ffg) - (fGg) = \int_a^b (\bar{g}f^{[n]} - f\overline{g^{\{n\}}})\,dx = [fg]_b - [fg]_a,$$

where

$$[fg]_x = i\sum_{k=1}^{n} f^{[n-k]}(x)\overline{g^{\{k-1\}}(x)}.$$

6) D_H is the set of all function $h(x)$ in D_F such that $[h,g]=0$ for every function g in D_G.

7) H is the transformation defined in the domain D_H which transforms $h(x)$ into $h^{[n]}(x)$: $Hh = h^{[n]}$, $(Hh, g) = (h, Gg)$.

8) D_T is the set of all functions $t(x)$ in D_G such that $[f,t]=0$ for every function f in D_F.

9) T is the transformation defined in the domain D_T which transforms $t(x)$ into $t^{\{n\}}(x)$: $Tt = t^{\{n\}}$, $(Ff, t) = (f, Tt)$.

Using the existence theorem of the quasi-differential equation $f^{[n]}(x) = f^*(x)$, $a' \leqslant x \leqslant b'$, $f^*(x) \subset L_2(a,b)$, it is easy to show that the transformations F, G, H, T are linear closed and $F^* \equiv T$, $G^* \equiv H$, $H^* \equiv G$, $T^* \equiv F$, where $*$ means the adjoint transformation.

Let $P(N)$, $C(N)$, $Q(N)$ and $R(N)$ be the point-spectrum, the continuous spectrum, the residual spectrum and the resolvent set respectively of an arbitrary linear transformation N[1]. Then

$$P(H) \subseteq P(F),\quad C(H) \subseteq P(F)+C(F),\quad Q(F) \subseteq Q(H),\quad R(F) \subseteq Q(H)+R(H),$$

$$P(T) \subseteq P(G),\quad C(T) \subseteq P(G)+C(G),\quad Q(G) \subseteq Q(T),\quad R(G) \subseteq Q(T)+R(T),$$

$$P(H) \subseteq \overline{P}(G)+\overline{Q}(G),\quad Q(H) \subseteq \overline{P}(G),\quad P(H)+Q(H) = \overline{P}(G)+\overline{Q}(G),$$

$$P(T) \subseteq \overline{P}(F)+\overline{Q}(F),\quad Q(T) \subseteq \overline{P}(F),\quad P(T)+Q(T) = \overline{P}(F)+\overline{Q}(F),$$

$$\overline{P}(F) \subseteq P(T)+Q(T),\quad \overline{Q}(F) \subseteq P(T),\quad C(F) = C(T),\quad R(F) = R(T),$$

$$\overline{P}(G) \subseteq P(H)+Q(H),\quad \overline{Q}(G) \subseteq P(H),\quad \overline{C}(G) = C(H),\quad \overline{R}(G) = R(H).$$

The necessary and sufficient condition that the residual spectrums of these transformations be empty sets is

$$P(H) = P(F) = \overline{P}(T) = \overline{P}(G).$$

If the functions $P_{k,\nu}(x)$ satisfy the conditions

$$\overline{P}_{n-k,\,n-k} = P_{k,k},\ k=0,1,\dots\left[\frac{n}{2}\right],\ \overline{\left(\frac{P_{n-\nu,\,n-k},\,P_{n-k,\,n-k}}{P_{n-\nu,\,n-\nu}}\right)} =$$

$$= P_{k,\nu},\ \nu < k,\ k=1,2,\dots n,\ \nu=0,1,\dots n-1,$$

where $\left[\frac{n}{2}\right]$ is the integer part of $\frac{n}{2}$, then $F \equiv G$, $H \equiv T$, $H^* \equiv F$, $F^* \equiv H$ and H is an Hermitian symmetric transformation. Denote by (p, q) the deficiency index of the transformation H. In accordance with the results of research of the solutions of the equation $Ff - lf = 0$, $I(l) \neq 0$, it follows that p and q can be equal only to one of the numbers 0, $n-2\left[\frac{n}{2}\right]$, $\left[\frac{n}{2}\right]$, $n-\left[\frac{n}{2}\right]$, n, independently from one another. Suppose that $p \leqslant q$ and denote by H_p the maximal symmetric extension of H. The function $f(x)$ belongs to D_{H_p}, if and only if it can be expressed in the form:

$$f(x) = h(x) + \sum_{k=1}^{p} c_k v_{p,k}(x),$$

where $h(x) \subset D_H$; c_k are arbitrary constants; $v_{p,k}(x) = f_k^+(x) - \sum_{j=1}^{p} a_{k,j} f_j^-(x)$, $f_k^+(x)$, $k=1,2,\dots p$, and $f_j^-(x)$, $j=1,2,\dots q$, are solutions of the equations $Ff - lf = 0$, $Ff - \bar{l}f = 0$, $I(l) \neq 0$, respectively, consisting a biorthonormal set; $\|a_{k,j}\|$ is a unitary matrix of the p th order. The function $g(x)$ belongs to $D_{H_p^*}$, if and only if $[v_{p,k}, g] = 0$, $k=1,2,\dots p$. The deficiency index of the transformation H_p is $(0,\ q-p)$.

If n is an even number and the transformation F is real, that is $\overline{Ff} \equiv F\bar{f}$, then $p=q$ can be equal only to one of the numbers 0, $\frac{n}{2}$, n, and H_p is a self-adjoint transformation (hypermaximal). All the points l, $I(l) \neq 0$, are in $R(H_p)$ and $Q(H_p)$ is an empty set. Denote by $f_k(x, l)$, $k=1,2,\dots 2m$, the orthonormal set of the solutions of the equation

$$f^{[n]} - lf = 0,\ I(l) \neq 0.$$

If $p=2m$, then all the functions $f_k(x, l)$, $k=1,2,\dots n$, are in $L_2(a, b)$. If $p=m$, then m functions, for instance, $f_1, f_2, \dots f_m$ are in $L_2(a, b)$ and m functions $f_{m+1}, f_{m+2}, \dots f_{2m}$ are either in $L_2(a, c)$ or in $L_2(c, b)$ where c is an arbitrary point interior to (a, b). If $p=0$, then m functions, for instance $f_1, f_2, \dots f_m$, are in $L_2(a, c)$ and m functions $f_{m+1}, f_{m+2}, \dots f_{2m}$ are in $L_2(c, b)$. The resolvent of the self-adjoint transformation H_p is an integral operator with the kernel $G(x, y; l, p)$:

$$G(x,y;l,n) = \frac{\begin{vmatrix} f_1(x,l) & f_2(x,l) \dots f_n(x,l) & g_n(x,y;l) \\ [f_1, v_{n,1}] & [f_2, v_{n,1}] \dots [f_n, v_{n,1}] & [g_n, v_{n,1}] \\ [f_1, v_{n,2}] & [f_2, v_{n,2}] \dots [f_n, v_{n,2}] & [g_n, v_{n,2}] \\ \dots & \dots & \dots \\ [f_1, v_{n,n}] & [f_2, v_{n,n}] \dots [f_n, v_{n,n}] & [g_n, v_{n,n}] \end{vmatrix}}{\begin{vmatrix} [f_1, v_{n,1}] & [f_2, v_{n,1}] \dots [f_n, v_{n,1}] \\ [f_1, v_{n,2}] & [f_2, v_{n,2}] \dots [f_n, v_{n,2}] \\ \dots & \dots \\ [f_1, v_{n,n}] & [f_2, v_{n,n}] \dots [f_n, v_{n,n}] \end{vmatrix}}$$

where

$$g_n(x,y;l) = \operatorname{sign}(x-y)\,\frac{1}{2}\sum_{k=1}^{n}\frac{1}{iP_{n,n}(y)}\,\frac{\partial \ln W(y,l)}{\partial f_k^{[n-1]}(y,l)}\,f_k(x,l),$$

$$W(x,l)=\begin{vmatrix} f_1^{[0]}(x,l) & f_2^{[0]}(x,l) & \dots f_n^{[0]}(x,l) \\ f_1^{[1]}(x,l) & f_2^{[1]}(x,l) & \dots f_n^{[1]}(x,l) \\ \dots & \dots & \dots \\ f_1^{[n-1]}(x,l) & (f_2^{[n-1]}(x,l) & \dots f_n^{[n-1]}(x,l) \end{vmatrix},$$

and where $[g_n, v_{n,k}]$ is calculated as a function of x.

$$G(x,y;l,m)=\frac{\begin{vmatrix} f_1(x,l) & f_2(x,l) & \dots & f_m(x,l) & g_m(x,y;l) \\ [f_1, v_{m,1}]_b & [f_2, v_{m,1}]_b & \dots & [f_m, v_{m,1}]_b & [g_m, v_{m,1}]_b \\ [f_1, v_{m,2}]_b & [f_2, v_{m,2}]_b & \dots & [f_m, v_{m,2}]_b & [g_m, v_{m,2}]_b \\ \dots & \dots & \dots & \dots & \dots \\ [f_1, v_{m,m}]_b & [f_2, v_{m,m}]_b & \dots & [f_m, v_{m,m}]_b & [g_m, v_{m,m}]_b \end{vmatrix}}{\begin{vmatrix} [f_1, v_{m,1}]_b & [f_2, v_{m,1}]_b & \dots [f_m, v_{m,1}]_b \\ [f_1, v_{m,2}]_b & [f_2, v_{m,2}]_b & \dots [f_m, v_{m,2}]_b \\ \dots & \dots & \dots \\ [f_1, v_{m,m}]_b & [f_2, v_{m,m}]_b & \dots [f_m, v_{m,m}]_b \end{vmatrix}},$$

where

$$g(x,y;l)=\begin{cases} -\sum\limits_{k=1}^{m} \frac{1}{iP_{n,n}(y)} \frac{\partial \ln W(y,l)}{\partial f_k^{[n-1]}(y,l)} f_k(x,l), & \text{for } x<y, \\ \sum\limits_{k=m+1}^{2m} \frac{1}{iP_{n,n}(y)} \frac{\partial \ln W(y,l)}{\partial f_k^{[n-1]}(y,l)} f_k(x,l), & \text{for } x>y, \end{cases}$$

and it is supposed that $f_{m+1}, f_{m+2}, \dots f_{2m}$ are in $L_2(c,b)$.

$$G(x,y;l,0)=\begin{cases} -\sum\limits_{k=1}^{m} \frac{1}{iP_{n,n}(y)} \frac{\partial \ln W(y,l)}{\partial f_k^{[n-1]}(y,l)} f_k(x,l), & \text{for } x<y, \\ \sum\limits_{k=m+1}^{2m} \frac{1}{iP_{n,n}(y)} \frac{\partial \ln W(y,l)}{\partial f_k^{[n-1]}(y,l)} f_k(x,l), & \text{for } x>y. \end{cases}$$

$G(x,y;l,n)$ is a kernel of Hilbert-Schmidt type. The real and imaginary parts of $G(x,y;l,m)$ and $G(x,y;l,0)$ are real kernels of Carleman type. If we denote by $E_p(\lambda)$ the resolutions of the identity of H_p, then the transformation $E_p(\Delta)=E_p(\beta)-E_p(\alpha)$, $\alpha \leqslant \lambda \leqslant \beta$, is an integral operator with kernel $E_p(x,y;\Delta)$ of Carleman type [1]. For fixed Δ and almost all y on (a,b) $E_p(x,y;\Delta)$ is a function of x in D_{H_p} and satisfies the equation

$$H_p E_p(x,y;\Delta)=\int_{\alpha}^{\beta} \mu \, d E_p(x,y;\Delta_\mu).$$

The kernel $E_p(x,y;\Delta)$ can be expressed as a Stieltjes integral

$$E_p(x,y;\Delta)=\sum_{k=1}^{n}\sum_{j=1}^{n} f_k(x,c;\lambda) f_j(y,c;\lambda) \, d\rho_{kj}(\lambda),$$

where $f_k(x,c;\lambda)$, $k=1,2,\dots n$, is the solution of the equation $f^{[n]}-\lambda f=0$ determined by the initial conditions $f_k^{[\nu-1]}(c,c;\lambda)=1$ or 0 according to $k=\nu$ or $k\neq\nu$; $\rho_{kj}(\lambda)$ is a function which is of bounded variation over every finite interval and which is continuous on the right. For a fixed value c, the functions $\rho_{kj}(\lambda)$ are uniquely determined apart from additive constants.

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MATHEMATICS

SOME RESULTS ON WARING'S PROBLEM FOR SMALLER POWERS

By LOO-KENG HUA

(Communicated by I. M. Vinogradow, Member of the Academy, 5. II. 1938)

Let k be a positive integer and $c_1, c_2, \ldots$ are positive integers depending on k only. Let $P(x)$ be an integral-valued polynomial of the kth degree with a positive first coefficient. Let $r_s(N)$ be the number of solution of

$$\sum_{i=1}^{s} P(x_i) = N, \quad x_i \geqslant 0; \tag{1}$$

and $R_s(N)$ be the number of solutions of

$$\sum_{i=1}^{s} p_i^k = N$$

where p's are primes.

What I am enabled to prove are the following theorems.

Theorem 1. Hardy-Littlewood's asymptotic formula (1) for $r_s(N)$ holds for $s \geqslant 2^k + 1$.

Theorem 2. Vinogradow's asymptotic formula (2) for $R_s(N)$ holds for $s \geqslant 2^k + 1$.

The proof of theorem 2 depends essentially on the following.

Lemma 1. Let

$$f(\alpha) = \sum_{x=1}^{P} e^{2\pi i x^k \alpha}, \quad P^k = N,$$

then

$$\int_0^1 |f(\alpha)|^{2^k} d\alpha = O(P^{2^k - k} (\log P)^{c_3}).$$

Or, more generally, for $m \leqslant k$,

$$\int_0^1 |f(\alpha)|^{2^m} d\alpha = O(P^{2^m - m} (\log P)^{c_3}).$$

The proof of lemma 1 is based on the following lemmas.

Lemma 2. Let $s \geqslant r$ and $n \ll P$. Then the number of solutions of

$$x_1^k + \ldots + x_r^k - x_{r+1}^k - \ldots - x_s^k \equiv O \pmod{n}, \quad x_\nu \ll P$$

is

$$\ll \frac{d^{c_3}(n)}{n} \sum_{n_1/n} \frac{(n, n_1^k)}{n_1} P^s (\log P)^{c_4}.$$

Lemma 3.

$$\sum_{m_1}^{P} \cdots \sum_{m_{c_3}}^{P} \frac{d^{c_4}(m)}{m} \sum_{n_1/m} \frac{(m, n_1^k)}{n_1} = O((\log P)^{c_7})$$

where m is the least common multiple of $m_1, \ldots, m_{c_3}$ and m_i runs over $O(P)$ positive numbers.

Some interesting consequences of theorem 2 are:

1. Every sufficiently large odd integer is a sum of nine cubes of primes.

2. Every sufficiently large integer $\equiv 17 \pmod{2^4 \cdot 3 \cdot 5}$ is a sum of 17 fourth powers of primes.

Certainly by the weapon used here, we are enable to prove that

3. Almost all odd integer are sums of five cubes of primes.

Here we state some results connected with theorem 1.

Let $P(x)$ satisfy the condition that there does not exist an integer $q\,(>1)$ such that $P(x) \equiv P(O) \pmod q$. Let $G(P(x))$ be the smallest integer s for which (1) is solvable for all sufficiently large N. Then we have:

1. For $k=4$

$$G(P(x)) \leqslant 17.$$

This contains Estermann, Davenport and Heilbronn's [3] result.

2. For $k=5$

$$G(P(x)) \leqslant 31.$$

The equality holds only when

$$P(x) \equiv aF(x) + b \pmod{2^5},\ 2 \nmid a \qquad (2)$$

where

$$F(x) = 2^5 F_5(x) - 2^4 F_4(x) + 2^3 F_3(x) - 2^2 F_2(x) + F_1(x)$$

and

$$F_i(x) = \frac{1}{i!}\, x(x-1)\ldots(x-i+1).$$

More definitely, by introducing Vinogradow's idea, we can prove that if $P(x)$ does not satisfy (2), then

$$G(P(x)) \leqslant 29.$$

The result 1 and 2 are improvements of my previous general result

$$G(P(x)) \leqslant 2^{k+1}(k-1)$$

which was obtained eight months ago.

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MATHÉMATIQUES

SUR UN THÉORÈME DE LA THÉORIE DES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

Par N. LUSIN, de l'Académie

S. Finikoff a attiré récemment l'attention sur l'intérêt qu'il y a à démontrer en toute rigueur une proposition relative aux systèmes d'équations simultanées aux dérivées partielles. Cette proposition ne paraît pas, peut-être, avoir un très grand intérêt théorique en soi; pratiquement, elle est beaucoup plus importante. Il suffit de signaler dans les diverses questions de l'analyse et de la géométrie son rôle d'un instrument transitoire dont l'utilité actuelle n'est pas négligeable (comme nous allons d'ailleurs en donner des exemples) et qui devra être regardé comme un moyen d'établir les existences. C'est ce que nous allons constater en donnant une solution complète d'une question de la géométrie différentielle classique posée, il y a déjà longtemps, dans la théorie des surfaces et dont la réponse paraissait présenter de difficultés sensibles.

Théorème. *Etant donné un système d'équations aux dérivées partielles mis sous la forme normale de Cauchy-Kowalewska*:

$$\frac{\partial^{r_1} U_1}{\partial x_1^{r_1}} = \Phi_1, \quad \frac{\partial^{r_2} U_2}{\partial x_1^{r_2}} = \Phi_2, \ldots, \quad \frac{\partial^{r_m} U_m}{\partial x_1^{r_m}} = \Phi_m, \tag{1}$$

$\Phi_1, \Phi_2, \ldots, \Phi_m$ *étant des fonctions holomorphes des variables indépendantes* $x_1, x_2, \ldots, x_n$ *des fonctions inconnues* $U_1, U_2, \ldots, U_m$ *et de leurs dérivées partielles jusqu'à l'ordre* $r_1, r_2, \ldots, r_m$ *respectivement (excepté les dérivées partielles qui figurent aux premiers membres) traitées comme des variables indépendantes, on peut trouver un système fixe de polynômes* $P_1, P_2, \ldots, P_m$ *en* $x_1, x_2, \ldots, x_n$ *à coefficients entiers tel que l'on ait les inégalités simultanées rigoureuses*:

$$|P_1 - U_1| > 0, \quad |P_2 - U_2| > 0, \ldots, |P_m - U_m| > 0,$$

dans un domaine D à n dimensions dépendant de $U_1, U_2, \ldots, U_m$, *et cet quelle que soit une solution holomorphe* $U_1, U_2, \ldots, U_m$ *du système donné* (1) *d'équations aux dérivées partielles.*

Démonstration. On sait que tout système d'équations aux dérivées partielles mis sous la forme normale de Cauchy-Kowalewska (1) peut être écrit sous la forme linéaire homogène:

$$\frac{\partial U_i}{\partial x_1} = \sum_{k,\, l} A_{kl}^{i} \frac{\partial U_k}{\partial x_l} \qquad \begin{matrix} (i, k = 1, 2, \ldots, \mu), \\ (l = 2, 3, \ldots, n), \end{matrix} \tag{2}$$

où les coefficients A^i_{kl} désignent les fonctions holomorphes des seules variables $U_1, U_2, \ldots, U_\mu$. Il importe de remarquer qu'on obtient la réduction du système donné (1) à la forme linéaire homogène (2) en faisant introduire de fonctions inconnues auxiliaires qui, jointes aux précédentes, constituent le système de fonctions $U_1, U_2, \ldots, U_\mu$, $\mu \geqslant m$, vérifiant le système réduit (2). On peut donc se borner à démontrer le théorème proposé dans le cas des systèmes d'équations linéaires homogènes (2).

Or, si nous prenons les conditions initiales:

$$U_1 = \omega_1,\ U_2 = \omega_2, \ldots,\ U_\mu = \omega_\mu \text{ pour } x_1 = x_1^0,$$

$\omega_1, \omega_2, \ldots, \omega_\mu$, étant des fonctions arbitraires de $x_2, x_3, \ldots, x_n$ développables par la série de Taylor au voisinage d'un point $x_2^0, \ldots, x_n^0$, les fonctions cherchées $U_1, U_2, \ldots, U_\mu$ devant être développables suivant les puissances de $x_1 - x_1^0, \ldots, x_n - x_n^0$ seront de la forme:

$$U_i(x_1, x_2, \ldots, x_n) = \omega_i + \frac{\omega_{i1}}{1!}(x_1 - x_1^0) + \frac{\omega_{i2}}{2!}(x_1 - x_1^0)^2 + \ldots + \frac{\omega_{ik}}{k!}(x_1 - x_1^0)^k + \ldots, \tag{3}$$

où ω_{ik} est la dérivée partielle $\frac{\partial^k U_i}{\partial x_1^k}$ calculée pour $x_1 = x_1^0$.

D'autre part, si nous prenons les dérivées successives $\frac{\partial}{\partial x_1}, \frac{\partial^2}{\partial x_1^2}, \frac{\partial^3}{\partial x_1^3}, \ldots$ de chacune des équations du système (2), en éliminant au moyen de ces équations et des équations obtenues les dérivées $\frac{\partial}{\partial x_1}, \frac{\partial^2}{\partial x_1^2}, \frac{\partial^3}{\partial x_1^3}, \ldots$ des fonctions inconnues $U_1, U_2, \ldots, U_\mu$, nous serons amenés aux équations:

$$\frac{\partial^k U_i}{\partial x_1^k} = \Omega_{ik}\left(U_1, \ldots, U_\mu, \frac{\partial U_1}{\partial x_2}, \ldots, \frac{\partial^k U_\mu}{\partial x_n^k}\right) \qquad (k = 1, 2, 3, \ldots), \tag{4}$$

dans lesquelles les Ω_{ik} désignent des fonctions des seules variables $U_1, U_2, \ldots, U_\mu$ et de leurs dérivées partielles jusqu'à l'ordre k prises par rapport aux variables $x_2, x_3, \ldots, x_n$. Il est à remarquer que les Ω_{ik} sont des fonctions holomorphes des lettres $U_1, U_2, \ldots, U_\mu$ et sont des polynômes en leurs dérivées partielles par rapport aux $x_2, x_3, \ldots, x_n$. En faisant $x_1 = x_1^0$ dans les équations (4), on aura les égalités:

$$\omega_{ik} = \Omega_{ik}\left(\omega_1, \ldots, \omega_\mu\ \frac{\partial \omega_1}{\partial x_2}, \ldots, \frac{\partial^k \omega_\mu}{\partial x_n^k}\right) \qquad \begin{matrix}(i = 1, 2, 3, \ldots, \mu), \\ (k = 1, 2, 3, \ldots).\end{matrix} \tag{5}$$

Ceci étant, prenons pour i un nombre fixe quelconque, $1 \leqslant i \leqslant \mu$, et faisons $k = 1, 2, 3, \ldots, \nu$ dans les égalités (5). Si nous joignons à ces égalités leurs dérivées partielles successives par rapport aux variables $x_2, x_3, \ldots, x_n$ jusqu'à l'ordre ν, il arrivera nécessairement un moment ν_0, où le nombre des équations obtenues surpassera celui des fonctions ω et de leurs dérivées partielles qui y figurent. En effet, lorsque nous prenons les dérivées partielles d'ordre $\leqslant \nu$ des égalités (5), $1 \leqslant k \leqslant \nu$, nous obtenons $\nu \cdot \frac{n(n+1)\ldots(n+\nu-1)}{1 \cdot 2 \cdot 3 \ldots \nu}$ d'égalités nouvelles et anciennes; d'autre part, les fonctions $\omega_1, \omega_2, \ldots, \omega_\mu$ et leurs dérivées partielles qui y figurent sont en nombre $\mu \cdot \frac{n(n+1)\ldots(n+2\nu-1)}{1 \cdot 2 \cdot 3 \ldots (2\nu)}$. Or, ce nombre sera évidemment inférieur au précédent, dès que ν surpassera le produit $\mu \cdot 2^n$.

À partir de ce moment ν_0, le nombre des égalités obtenues surpassera celui des fonctions arbitraires ω et de leurs dérivées partielles. Comme les Ω_{ik} sont des fonctions holomorphes des variables ω et sont des polynômes en leurs dérivées partielles, il existe nécessairement au moins une fonction holomorphe $\mathfrak{F}(t_1, t_2, \ldots, t_s)$ des variables indépendantes $t_1, t_2, \ldots, t_s$ qui n'est pas identiquement nulle et telle que si nous remplaçons les variables t par les quantités ω_{ik}, $(k=1,2,3,\ldots,\nu_0)$, et par leurs dérivées partielles jusqu'à l'ordre ν_0, prises par rapport aux variables $x_2, x_3, \ldots, x_n$, nous obtiendrons une expression identiquement nulle, quelles que soient les fonctions initiales arbitraires $\omega_1, \omega_2, \ldots, \omega_\mu$.

Cela posé, soit $t_1^0, t_2^0, \ldots, t_s^0$ un système quelconque de valeurs des variables $t_1, t_2, \ldots, t_s$ pour lequel la fonction $\mathfrak{F}$ est holomorphe et sûrement non nulle. Comme tout polynôme en $x_2, x_3, \ldots, x_n$ est parfaitement déterminé par sa valeur et par les valeurs de ses dérivées partielles calculées en un point fixe $x_2^0, x_3^0, \ldots, x_n^0$ et prises arbitrairement, nous pouvons remplacer les quantités ω_{ik} par des polynômes p_{ik} en $x_2, x_3, \ldots, x_n$ de manière que le résultat de la substitution des p_{ik} et de leurs dérivées partielles dans la fonction $\mathfrak{F}$ soit sûrement non nul pour le point fixe $x_2^0, x_3^0, \ldots, x_n^0$. Dans ces conditions, on voit bien que le système des polynômes P_i en $x_1, x_2, x_3, \ldots, x_n$ définis par les égalités:

$$P_i = p_i + \frac{p_{i1}}{1!}(x_1 - x_1^0) + \frac{p_{i2}}{2!}(x_1 - x_1^0)^2 + \ldots + \frac{p_{i\nu_0}}{\nu_0!}(x_1 - x_1^0)^{\nu_0} \quad (i=1,2,3,\ldots\mu), \tag{6}$$

vérifie l'énoncé du théorème proposé quels que soient les polynômes p_i en $x_2, x_3, \ldots, x_n$.

Il ne reste qu'à montrer que les coefficients des polynômes P_i peuvent être pris parmi les nombres entiers. Or, d'après les recherches de M. I. Chlodovsky, toute fonction holomorphe $f(x_2, x_3, \ldots, x_n)$ peut être approchée, ainsi que ses dérivées partielles, jusqu'à un ordre assigné d'avance, par un polynôme $\pi(x_2, x_3, \ldots, x_n)$ à coefficients entiers et par ses dérivées partielles correspondantes dans tout domaine D qui ne contient aucun point $x_2, x_3, \ldots, x_n$ à une coordonnée entière, cette approximation pouvant être faite aussi précise que l'on veut. Il en résulte que nous pouvons remplacer, dans les égalités (6), les polynômes $\frac{p_{i1}}{1!}, \ldots, \frac{p_{i\nu_0}}{\nu_0!}$ par les polynômes $q_{i1}, \ldots, q_{i\nu_0}$ à coefficients entiers de manière que le résultat de la substitution dans $\mathfrak{F}$ des $1!\, q_{i1}, \ldots; \nu_0!\, q_{i\nu_0}$ et de leurs dérivées partielles soit encore non nul pour un point fixe $x_2^0, x_3^0, \ldots, x_n^0$. Il est évident qu'en prenant pour p_i des polynômes quelconques en $x_2, x_3, \ldots, x_n$ à coefficients entiers, nous achevons la démonstration du théorème proposé.

Remarque. Nous avons supposé que la solution $U_1, U_2, \ldots, U_m$ du système (1) est holomorphe. Or, le raisonnement s'étend immédiatement aux solutions $U_1, U_2, \ldots, U_m$ qui, sans être analytiques, possèdent les dérivées partielles continues jusqu'à l'ordre ν_0 (dépendant des seuls entiers positifs $r_1, r_2, \ldots, r_m, n$). En effet, les équations (4) et les égalités (5) restent intactes ainsi que la détermination de la fonction $\mathfrak{F}$. Dans ces conditions, les polynômes (6) vérifient encore l'énoncé du théorème proposé. Pour réduire d'avantage le nombre des dérivées partielles des fonctions U_i nous avons besoin d'une autre méthode.

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MÉCANIQUE CÉLESTE

SUR LA CONVERGENCE DES SÉRIES REPRÉSENTANT FORMELLEMENT DES SOLUTIONS SIMPLE-PÉRIODIQUES

Par N. MOISSEIEV

(Présenté par V. G. Fesenkov, de l'Académie, le 23.I.1938)

Soient

$$\left.\begin{aligned} \ddot{r} &= r\dot{\vartheta}^2 + 2nr\dot{\vartheta} + U_r', \\ \ddot{\vartheta} &= -\frac{2\dot{r}\dot{\vartheta}}{r} - 2n\frac{\dot{r}}{r} + \frac{1}{r^2}U_\vartheta', \\ n &= \text{const}, \quad U = U(r, \vartheta), \end{aligned}\right\} \tag{1}$$

les équations différentielles d'un problème de la mécanique céleste. On connaît plusieurs méthodes pour la recherche des séries de la forme:

$$r = a_0 + \sum_{k=1}^{\infty}(a_k \cos k\vartheta + b_k \sin k\vartheta), \tag{2}$$

représentant formellement des solutions simple-périodiques du problème (1). Le but de cette note est de faire quelques remarques concernant la convergence des séries (2); nous la commencerons par l'indication d'une méthode à obtenir ces séries; il nous semble que cette méthode n'a pas été publiée jusqu'à présent. Soit

$$f \equiv r - \Phi(\vartheta) = 0 \tag{3}$$

l'équation d'une courbe analytique. La caractéristique des contacts des trajectoires du problème (1), appartenant à une famille isoénergétique (h), avec cette courbe (3) sera [1, 2]

$$\ddot{f}_\pm = \frac{1}{P^2}\left[2(U+h)S \pm 2nP^3\sqrt{2(U+h)} + P^2Q\right], \tag{4}$$

où

$$\left.\begin{aligned} P^2 &= \frac{1}{r^2}\left(r^2 + \Phi'^2\right), \\ Q &= \frac{1}{r^2}\left(U_r' r^2 - U_\vartheta'\Phi'\right), \\ S &= \frac{1}{r^3}\left(r^2 + 2\Phi'^2 - r\Phi''\right), \end{aligned}\right\} \tag{5}$$

et où r est exprimé en fonction de ϑ à l'aide de l'équation (3). Des deux signes devant le radical dans la formule (4) il faut choisir le signe

supérieur (plus) pour le cas des mouvements directs et le signe inférieur (moins) pour le cas des mouvements rétrogrades.

Si la caractéristique des contacts $j_+(j_-)$ le long de la courbe (3) devient identiquement nulle, la courbe (3) elle-même est nécessairement une trajectoire du problème (1), correspondant à la direction du mouvement direct (rétrograde). Alors la condition pour que la courbe (3) coïncide avec l'une des trajectoires du problème (1) peut être écrite dans la forme:

$$2(U+h)S \pm 2nP^3\sqrt{2(U+h)}+P^2Q=0, \tag{6}$$

ou bien dans la forme

$$\begin{aligned} &4(U+h)^2r^2(r^2+2\Phi'^2-r\Phi'')^2+ \\ &+4(U+h)r(r^2+\Phi'^2)[(r^2+2\Phi'^2-r\Phi'')(U'_r r^2-U'_\vartheta\Phi')+ \\ &-2n^2r(r^2+\Phi'^2)^2]+(r^2+\Phi'^2)^2(r^2U'_r-\Phi'U'_\vartheta)^2=0, \end{aligned} \tag{7}$$

où r doit être exprimé en fonction de ϑ à l'aide de l'équation (3). Cette équation (7) peut être envisagée comme une équation différentielle de la trajectoire du problème (1), appartenant à une famille isoénergétique (h). En vertu de la disparition du signe double, cette équation détermine les trajectoires directes aussi bien que les trajectoires rétrogrades. La distinction peut être faite à l'aide de la forme (6) de la même équation.

En remplaçant dans l'équation (7) la fonction $\Phi(\vartheta)$ par la série (2), nous pouvons obtenir un système infini d'équations, déterminant les coefficients a_0, a_k et b_k en fonction de la constante de Jacobi h. Dans le cas général ces fonctions ne seront pas uniformes. Par cette raison la méthode précédente, ainsi que toute autre méthode de la recherche des séries formelles, nous fournira, pour un h donné, un certain nombre $N_+(h)$ de séries (2), représentant formellement les solutions périodiques, faisant une simple révolution dans la direction directe autour du point $r=0$, ainsi qu'un certain nombre $N_-(h)$ de séries analogues, correspondant aux solutions simple-périodiques rétrogrades. Pour être brefs, bornons nous aux trajectoires directes et supposons que le nombre $N_+(h)$ est un nombre fini.

Soit $N^c_+(h)$ le nombre des séries convergentes qui se trouvent parmi les $N_+(h)$ séries formelles que nous avons obtenues. Il est évident que $N^c_+(h)$ doit aussi représenter le nombre des trajectoires diverses, appartenant à la famille (h), qui sont fermées et qui font une simple révolution dans le sens direct autour du point $r=0$. Les autres séries, dont le nombre est $N_+(h)-N^c_+(h)$, ne sont que des séries divergentes, ne représentant aucune solution de notre problème.

En supposant que le nombre $N_+(h)$ est connu, nous voulons dans ce qui suit étudier la question sur la détermination du nombre $N^c_+(h)$. Comme point d'appui nous prenons un théorème démontré dans un article précédent [2, 3] et lié au critère d'existence des trajectoires périodiques de M. Whittaker. Admettons que l'étude des contacts des trajectoires de la famille isoénergétique (h) avec les courbes d'un certain système topographique de cycles analytiques et réguliers, faisant chacun une simple révolution autour du point $r=0$, nous a donné la possibilité d'établir l'existence de $M_+(h)$ régions annulaires de M. Whittaker sans points communs, dont chacune—en vertu du théorème précédemment cité [2, 3]—contient au moins une trajectoire périodique et régulière, faisant une simple révolution dans le sens direct autour du point $r=0$.

Cela étant posé, nous pouvons énoncer les propositions suivantes:

Théorème 1. Le nombre $N_+^c(h)$ des séries convergentes (2) ne peut être moindre que le nombre $M_+(h)$ des régions de M. Whittaker.

Théorème 2. Si le nombre $N_+(h)$ des séries formelles (2) est égal au nombre $M_+(h)$ des régions annulaires de M. Whittaker, chacune des $N_+(h)$ séries formelles converge et représente une solution simple-périodique du problème (1). Chacune des régions de M. Whittaker contient dans ce cas une et une seule trajectoire simple-périodique. Enfin toutes les $N_+(h)$ trajectoires périodiques n'ont pas de points communs deux à deux.

Ces deux théorèmes peuvent être aussi appliqués au cas, où au lieu de l'ensemble de toutes les trajectoires simple-périodiques nous envisageons seulement une certaine partie de cet ensemble, contenant les trajectoires, qui possèdent une certaine propriété commune caractéristique (P). Soit $N_+(h, P)$ le nombre de toutes les solutions formelles (2), qui peuvent jouir de la propriété (P), est soit $M_+(h, P)$ le nombre des régions de M. Whittaker, jouissant de la même propriété. Cela étant posé, nous pouvons affirmer que les théorèmes 1 et 2 sont valables par rapport aux nombres $N_+(h, P)$ et $M_+(h, P)$ aussi bien, qu'elles étaient valables par rapport aux nombres $N_+(h)$ et $M_+(h)$.

Quant à la condition (P), elle peut prendre plusieurs formes diverses. On peut, par exemple, entendre sous la condition (P) — la condition de la convergence vers zéro de toutes les dimensions de l'orbite quand la constante de Jacobi h tend vers une certaine valeur limite h_0. Alors $N_+(h, P)$ sera le nombre des séries formelles (2), dont tous les coefficients a_0, a_k et b_k tendent vers zéro avec $h \to h_0$, tandis que $M_+(h, P)$ sera le nombre des régions de M. Whittaker, dont les dimensions tendent aussi vers zéro sous la même condition par rapport à h.

Il nous reste à donner quelques exemples de l'application de nos théorèmes.

Exemple 1. Le problème de trois corps de G. Hill.

Dans ce cas particulier du problème restreint de trois corps G. Hill a obtenu [4] des séries, représentant formellement les coordonnées rectangulaires x et y dans une orbite simple-périodique à mouvement synodique direct autour de la masse attirante T. Les coefficients de ces séries sont des fonctions uniformes de la constante de Jacobi h, tendant vers zéro quand h tend vers $-\infty$. En vertu de ces conditions nous avons dans ce cas $N_+(h, P) = 1$.

Mais, comme nous avons démontré [3], pour chaque h dans un certain intervalle $(-\infty, h_1)$ il existe pour le même problème une région de M. Whittaker, correspondant aux trajectoires du mouvement synodique direct et se réduisant à la masse attirante T pour h tendant vers $-\infty$. En conséquence nous avons ici $M_+(h, P) = 1$.

Alors, en vertu du théorème 2, nous pouvons affirmer, que les séries formelles de Hill sont convergentes pour chaque h d'un certain intervalle $(-\infty, h_2)$ où la valeur h_2 est au moins égale à h_1. En même temps nous sommes assurés, que pour chaque h de l'intervalle $(-\infty, h_1)$ à l'intérieur de la région annulaire de M. Whittaker correspondante il n'y a qu'une seule trajectoire simple-périodique à mouvement direct.

Ainsi nous avons démontré l'identité de la classe des orbites de G. Hill et de notre classe $(\ddot{\mathfrak{p}}_+^0)_d$ des orbites périodiques, identité dont nous avons souligné la vraisemblance dans l'article cité [3].

Quant à la convergence des séries de G. Hill, elle a été démontrée pour la première fois par A. Liapounoff [5], à l'aide de méthodes purement analytiques.

Exemple 2. Les orbites périodiques autour des points de libration collinéaires dans le problème restreint des trois corps.

Soit L un des points de libration nommés. Il est connu [6] que la recherche des séries formelles, représentant les solutions simple-périodiques du genre précédemment indiqué, nous amène à la conclusion que pour chaque valeur de h il existe une seule solution formelle de la forme (2), possédant la propriété que les dimensions de l'orbite tendent vers zéro, quand h tend vers sa valeur critique h_L. A cette solution formelle correspond la direction rétrograde du mouvement synodique. Donc nous avons $N_-(h,P)=1$.

Mais il est aisé de montrer, en faisant usage de la méthode de notre article [7], que pour chaque valeur de h d'un certain intervalle (h_L, h_1) il existe une seule région annulaire de M. Whittaker, contenant à son intérieur au moins une trajectoire simple-périodique, entourant le point L et correspondant au mouvement synodique rétrograde. Les dimensions de la région nommée tendent vers zéro, quand h tend vers la valeur h_L. Par cette raison nous avons $M_-(h,P)=1$.

Donc toutes les conditions du théorème 2 sont remplies. Il en suit que les séries formelles de A. Liapounoff convergent au moins pour h compris dans l'intervalle (h_L, h_1). Quant à nos régions annulaires, chacune d'elles ne contient qu'une seule trajectoire simple-périodique à mouvement rétrograde. La classe des orbites périodiques de A. Liapounoff est donc identique à notre classe (j^0_-) établie dans l'article cité [7].

La convergence des séries formelles de Liapounoff, dont nous venons de donner une nouvelle démonstration, a été établie pour la première fois par A. Liapounoff lui-même.

Nous terminons cette note par la remarque suivante: Il est connu que les brillantes recherches numériques de M. Strömgren [8] ont amené cet éminent astronome à la conclusion qu'il existe trois classes d'orbites simple-périodiques ayant pour origine les orbites infinitésimales rétrogrades autour des points de libration collinéaires L_1, L_2 et L_3 (les classes a, b et c). Il résulte de ce qui précède, que les origines de ces classes de M. Strömgren ne peuvent être autre chose que les classes correspondantes de A. Liapounoff précédemment citées.

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MÉCANIQUE

SUR LA STABILITÉ JACOBIENNE GÉNÉRALISÉE D'UNE TRAJECTOIRE PÉRIODIQUE

Par N. MOISSEIEV

(Présenté par V. G. Fesenkov, de l'Académie, le 23. I. 1938)

Soit T une trajectoire simple-périodique et régulière, à mouvement synodique direct * d'un problème dynamique

$$\ddot{x} = 2n\dot{y} + U'_x,$$
$$\ddot{y} = -2n\dot{x} + U'_y, \qquad (1)$$
$$n = \text{const}^{**};\ U = U(x, y),$$

correspondant à la valeur h_T de la constante h de l'intégrale de Jacobi:

$$\dot{x}^2 + \dot{y}^2 = 2(U + h). \qquad (2)$$

Soit

$$f(x, y) = f \qquad (3)$$

un système topographique de cycles simples et réguliers, défini pour toutes les valeurs du paramètre f de l'intervalle

$$f_i \leqslant f \leqslant f_e \qquad (4)$$

et possédant la propriété suivante: à chaque couple (f_1, f_2) des valeurs de f de cet intervalle, vérifiant l'inégalité

$$f_1 < f_2,$$

il correspond une couple de cycles (3)

$$f(x, y) = f_1 \quad \text{et} \quad f(x, y) = f_2$$

telle, que la courbe

$$f(x, y) = f_1$$

est entièrement comprise à l'intérieur de la courbe

$$f(x, y) = f_2.$$

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* Le cas du mouvement rétrograde peut être envisagé d'une manière tout à fait analogue.

** Dans le cas $n = 0$ toutes les remarques, concernant la direction du mouvement le long des trajectoires, sont inutiles.

Nous désignons par (f_i, f_e) la région annulaire du plan xy, limitée par les courbes:

$$f(x, y) = f_i, \tag{5}$$

et

$$f(x, y) = f_e, \tag{6}$$

et nous supposons que la fonction $f(x, y)$ admet partout à l'intérieur de l'anneau (f_i, f_e) des dérivées partielles continues du premier et du second ordre par rapport aux variables x et y. Nous supposons enfin que l'anneau (f_i, f_e) ne contient à son intérieur aucun des points singuliers de la fonction des forces $U(x, y)$, ainsi qu'aucun des points de la courbe des vitesses nulles.

Nous appelerons chaque anneau (f_i, f_e) de cycles (3), possédant toutes les propriétés indiquées,—un anneau admissible.

La trajectoire T étant donnée, il peut arriver que nous pourrons construire un anneau (4) admissible de cycles (3) tel, que T sera l'une des courbes (3), correspondant à une valeur f_T du paramètre f, vérifiant l'inégalité

$$f_i < f_T < f_e.$$

Dans ce cas nous dirons que la trajectoire T est englobée dans l'anneau (4) des cycles admissibles (3).

Après ces remarques préliminaires nous pouvons formuler la définition suivante de stabilité Jacobienne généralisée.

S'il est possible d'englober la trajectoire T dans un anneau (f_i, f_e) des cycles (3) admissibles, tel que la caractéristique des contacts [1] des trajectoires du problème (1), appartenant à la famille isoénergétique (h_T), avec les cycles (3), calculée pour la même direction du mouvement synodique que la direction de T, c'est-à-dire—la fonction $\ddot{f}_+(h_T)$*, est positive partout à l'intérieur de l'anneau (f_i, f_T), négative partout à l'intérieur de l'anneau (f_T, f_e), et ne s'annule en aucun point de l'anneau total (f_i, f_e), sauf les points de la trajectoire T, dans ce cas nous dirons que T possède la stabilité Jacobienne généralisée par rapport à l'anneau (f_i, f_e) des cycles admissibles. La notation abrégée de ce fait sera: T est stable dans le sens $SJG(f_i, f_e)$.

S'il est possible d'englober la trajectoire T dans un anneau (f_i, f_e) des cycles (3) admissibles, tel que la caractéristique des contacts $\ddot{f}_+(h_T)$** est négative partout à l'intérieur de l'anneau (f_i, f_T), positive partout à l'intérieur de l'anneau (f_T, f_e), et ne s'annule qu'aux points de T, dans ce cas nous dirons que T possède l'instabilité Jacobienne généralisée par rapport à l'anneau (f_i, f_e) des cycles admissibles. La notation abrégée de ce fait sera: T est instable dans le sens $IJG(f_i, f_e)$.

Désignons par δn le déplacement le long de la normale à la trajectoire T. Donc, le système des courbes

$$\delta n = c \tag{7}$$

défini dans un certain intervalle des valeurs de c:

$$-a \leqslant c \leqslant a, \tag{8}$$

où a est suffisamment petit, n'est qu'une espèce particulière de l'anneau admissible (3), (4). Cela nous montre [2] que la stabilité (l'instabilité)

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* Dans le cas, où T a la direction rétrograde, nous devons prendre la fonction $\ddot{f}_-(h_T)$.

** Voir la remarque précédente.

de la trajectoire périodique T au sens de Jacobi usuel n'est qu'un cas particulier de la stabilité (l'instabilté) Jacobienne généralisée, que nous avons définie.

Pour éviter tout malentendu possible, il faut noter que c'est la stabilité (l'instabilité) de la trajectoire périodique entière qui nous occupe ici et non la stabilité (l'instabilité) de certaines parties ou de certains points de cette trajectoire. En vertu de cela, nous nommons T stable au sens de Jacobi usuel [la notation abrégée SJ, ou bien $SJG(\delta n)$] seulement dans le cas, où la caractéristique D, entrant dans l'équation bien connue

$$\frac{d^2\delta n}{dt^2} = D \cdot \delta n, \tag{9}$$

est négative partout sur la trajectoire T et ne s'annule en aucun point de T. Quant à l'instabilité—nous nommons T instable au sens de Jacobi usuel [la notation abregée IJ, ou bien $IJG(\delta n)$] seulement dans le cas où la caractéristique D est positive partout sur T et ne s'annule en aucun point de T. Indiquons quelques propriétés des trajectoires du problème (1), qui sont liées avec la notion de stabilité Jacobienne généralisée, et qui peuvent être démontrées presque immédiatement en faisant usage de la théorie des caractéristiques des contacts [1].

1) S'il existe un anneau admissible (f_i, f_e), par rapport auquel T possède la stabilité (l'instabilité) Jacobienne généralisée, dans ce cas il est impossible de trouver un autre anneau admissible (g_i, g_e) par rapport auquel T possède l'instabilité (la stabilité) Jacobienne généralisée.

En vertu de cela la trajectoire T ne peut être stable et instable simultanément au sens de Jacobi généralisé.

2) Si T est stable au sens $SJG(f_i, f_e)$, il est impossible qu'elle soit instable au sens $IJG(\delta n)$. Si T est instable au sens $IJG(f_i, f_e)$, il est impossible qu'elle soit stable au sens $SJG(\delta n)$.

3) Si T est stable au sens SJ, elle est stable aussi au sens généralisé de Jacobi. Si T est instable au sens IJ, elle est aussi instable au sens généralisé de Jacobi.

4) Si T est stable au sens $SJG(f_i, f_e)$, chacune des trajectoires voisines, ayant des contacts avec les cycles (3) à l'intérieur de l'anneau (f_i, f_e), doit avoir des points communs avec T et d'ailleurs l'arc de la trajectoire voisine entre le point de contact et le point commun doit être compris dans l'anneau*. Par «arcs des trajectoires voisines» nous entendons ici les arcs des trajectoires, qui appartiennent à la même famille isoénergétique (h_T) que T, et qui vont dans l'intérieur de l'anneau (f_i, f_e) toujours dans la même direction du mouvement synodique que T**. En conséquence, la trajectoire voisine ne peut avoir de boucles, qui ne contiennent pas la courbe limite (5), et qui en même temps sont entièrement comprises à l'intérieur de l'anneau. En faisant usage de la terminologie abrégée, nous dirons aussi que la trajectoire voisine ne peut être une trajectoire de type «paghidale», ou bien «paghidale-périplégmatique», dont les «paghides» [les boucles qui n'entourent pas la courbe (5)] sont comprises à l'intérieur de l'anneau (f_i, f_e).

5) Si T est stable au sens $SJG(f_i, f_e)$, il n'existe point d'autres trajectoires simple-périodiques voisines, ne sortant pas de l'anneau (f_i, f_e) et ne coupant pas la trajectoire T.

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* Il est vraisemblable que les points communs nommés sont exclusivement des points d'intersection et que les trajectoires asymptotiques à T qui ne coupent pas T sont impossibles dans ce cas.

** Dans le cas $n = 0$ cette condition, ainsi que toutes les autres concernant la direction du mouvement, peut être omise.

6) Si T est instable au sens $IJG(f_i, f_e)$, aucune des trajectoires voisines, ayant des contacts à l'intérieur de l'anneau (f_i, f_e) avec les cycles (3), ne peut atteindre des points communs avec T, sans sortir d'abord de l'anneau.

7) Si T est instable au sens $IJG(f_i, f_e)$, il n'existe point d'autres trajectoires voisines, n'ayant pas de points en dehors de l'anneau (f_i, f_e).

8) Si T est instable au sens $IJG(f_i, f_e)$, toutes les semi-trajectoires positives voisines, qui ne sortent pas de l'anneau (f_i, f_e), doivent être asymptotiques à la trajectoire T.

L'expression «la semi-trajectoire positive» doit désigner ici la partie de la trajectoire, décrite par le point mobile quand le temps t croît indéfiniment en partant d'un certain moment initial t_0. Dans le cas de la trajectoire périodique la «semi-trajectoire» coïncide avec la trajectoire entière.

Sans prolonger l'exposé des diverses propriétés de trajectoires périodiques liées à la notion de stabilité Jacobienne généralisée, nous remarquons seulement, que la théorie de cette stabilité est d'une importance considérable pour la théorie de la localisation des trajectoires périodiques à l'aide des critères, semblables à celui de M. Whittaker [3]. Il est remarquable, que la notion de stabilité Jacobienne usuelle est moins intimement liée à la théorie nommée de la localisation, que la notion de stabilité généralisée.

Nous concluons cette note par un simple exemple de l'application des deux notions de stabilité Jacobienne en question.

Soit $n=0$ et $U=\frac{1}{r}$, où $r=\sqrt{x^2+y^2}$. Supposons que la constante h de l'intégrale (2) est négative. Dans ce cas tous les trajectoires du problème (1) sont des ellipses Keplériennes. Prenons une de ces ellipses, par exemple

$$r=\frac{p_T}{1-e_T\cos\varphi}$$

pour notre trajectoire T. La valeur correspondante de la constante h_T est liée à e_T et p_T par la formule

$$h_T=-\frac{(1-e_T^2)}{2p_T}.$$

Prenons le système topographique des ellipses:

$$p=r(1-e_T\cos\varphi).$$

La condition du contact s'écrira:

$$\dot{p}=\dot{r}(1-e_T\cos\varphi)+re_T\dot{\varphi}\sin\varphi=0.$$

La caractéristique du contact, c'est-à-dire la valeur de la seconde dérivée:

$$\ddot{p}=\ddot{r}(1-e_T\cos\varphi)+2\dot{r}\dot{\varphi}e_T\sin\varphi+r\dot{\varphi}^2e_T\cos\varphi+r\ddot{\varphi}e_T\sin\varphi,$$

calculée à l'aide des équations du mouvement, de l'intégrale d'énergie et de la condition du contact, en fonction des coordonnées polaires r et φ et de la constante h_T, aura la forme:

$$\ddot{p}=\frac{(1-e_T^2)(1-e_T\cos\varphi)^2}{r(1-2e_T\cos\varphi+e_T^2)}\left(\frac{1}{p}-\frac{1}{p_T}\right).$$

En supposant que $p_T > 0$, nous pouvons toujours choisir deux valeurs p_i et p_e de telle manière que

$$0 < p_i < p_T < p_e,$$

et que l'anneau (p_i, p_e) soit un anneau de cycles admissibles. L'expression pour $\ddot{p}$ nous montre que nous avons:

$$\ddot{p} > 0 \quad \text{pour} \quad p_i \leqslant p < p_T,$$

et

$$\ddot{p} < 0 \quad \text{pour} \quad p_T < p \leqslant p_e.$$

Il en résulte que l'ellipse T est stable au sens $SJG(p_i, p_e)$. Alors toutes les ellipses Keplériennes (sauf celles pour lesquelles $p_T = 0$) possèdent la stabilité Jacobienne généralisée du type $SJG(p_i, p_e)$ et ce fait est en parfait accord avec la remarque évidente, que toutes les ellipses isoénergétiques doivent s'entrecouper. Donc, cette remarque peut être envisagée (dans un certain sens) comme une des conséquences de notre résultat. Quant à la stabilité Jacobienne usuelle, elle ne peut pas nous fournir une conclusion d'une telle généralité. En effet, comme il est démontré par M. Stepanoff[4], ce sont seulement les ellipses pour lesquelles $\frac{p}{1+e} \geqslant -\frac{1}{4h}$, qui sont stables au sens de Jacobi usuel, tandis que sur chacune des autres ellipses la caractéristique D n'est point une fonction d'un signe invariable.

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PHYSIK

BORNSCHE ELEKTRODYNAMIK UND KOSMOLOGIE

Von H. MANDEL

(Vorgelegt von A. F. Joffe, Mitglied der Akademie, d. 2.II.1938)

I. Fünfdimensionale einheitliche Feldtheorie ermöglichte eine geometrische Begründung der klassischen Maxwellschen Elektrodynamik. Dabei wurde der der scg. Zylindrizitätsbedingung genügender fünfdimensionaler Raum $\overset{\circ}{R}_5$ betrachtet, in welchem eine Kongruenz $\overset{\circ}{X}^i$ vorausgesetzt wird, die den Gleichungen:

$$\overset{\circ}{\nabla}_i \overset{\circ}{X}_k + \overset{\circ}{\nabla}_k \overset{\circ}{X}_i = 0 \tag{1}$$

genügen muss. Born hat nun eine verallgemeinerte nichtlineare Elektrodynamik vorgeschlagen, welche ebenfalls eine klassische (d. h. keine Quanten-) Theorie darstellt*. Es liegt daher der Versuch nahe, die Idee der geometrischen Begründung in der Form einer fünfdimensionalen Theorie auf den Fall der Bornschen Elektrodynamik zu übertragen.

Die Lagrangesche Funktion des elektromagnetischen Feldes soll nach der Bornschen Theorie eine Quadratwurzel** $\sqrt{I}$ darstellen, deren Entwicklung in erster Nähung die Lagrangesche Funktion des Maxwellschen Feldes ergibt. Die Invariante I wird aus dem Tensor

$$F_{ik} = 2^{-1/2} b g_{ik} + \mathfrak{M}_{ik} \text{***} \tag{2}$$

(b — absolutes Feld, $\mathfrak{M}_{ik}$ — Tensor des elektromagnetischen Feldes) in der Form eines Quadrats

$$I = \frac{1}{2} F_{\alpha\beta} F^{\alpha\beta} \tag{3}$$

(oder in Form einer Determinante $|F_{ik}|$) gebildet. Der Tensor F_{ik} enthält einen symmetrischen (skalaren) Bestandteil und damit unterscheidet er sich

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* Es soll hier nicht diskutiert werden, inwiefern und ob die Bornsche Theorie als eine klassische Theorie physikalischen Sinn haben kann. Nichtdestoweniger glauben wir, dass man die klassische Bornsche Elektrodynamik, bei gewissen Voraussetzungen, als eine unvollkommene Beschreibung einiger elektrodynamischen Folgerungen der Theorie des Positrons ansehen darf.

** Unsere Betrachtung bezieht sich ebenfalls auf andere Gestalten der Lagrangeschen Funktion, die von Infeld untersucht wurden. Vgl. L. Infeld, Cambridge Proc., 33, 1, 70 (1937).

*** F_{ik} its hier als Vierertensor zu verstehen, ebenso wie g_{ik} (in den folgenden Formeln ist g_{ik} durchweg fünfdimensional).

vom Tensor des elektromagnetischen Feldes $\mathfrak{M}_{ik}$; deswegen unterscheidet sich auch (3) von der (doppelten) Lagrangeschen Funktion der Maxwellschen Theorie $\frac{1}{2}\mathfrak{M}_{\alpha\beta}\mathfrak{M}^{\alpha\beta}$. Die geometrische Deutung der Maxwellschen Gesetzen beruht auf der Identifizierung der Tensoren

$$\overset{\circ}{\nabla}_i \overset{\circ}{X}_k = \frac{\sqrt{-}}{2}\mathfrak{M}_{ik}^{*} \tag{4}$$

[man beachte (1)]. Die geometrische Deutung der Gesetze der Bornschen Theorie würde also in dem Falle möglich sein, wenn wir den Grundtensor der Bornschen Theorie F_{ik} in einer analogen Weise geometrisch einführen könnten.

Dazu untersuchen wir solcheinen 5-dimensionalen Raum R_5, in welchem eine Kongruenz X^i existiert, die statt (1) den Gleichungen:

$$\nabla_i X_k + \nabla_k X_i = g_{ik} \text{ skalar} \tag{5}$$

genügt, so dass die Identifizierung

$$\overline{\nabla_i X_k} = \frac{\sqrt{-}}{2} F_{ik}\sigma \; ^{**} \tag{6}$$

möglich wird. Die Bedeutung des skalaren Faktors σ wird im Folgenden sogleich klar gemacht; beim Übergang zu vierdimensionalen Formeln kann $\sigma = 1$ gesetzt werden.

Die Zylindrizitätsbedingung bedeutete die Konstanz des metrischen Tensors $\overset{\circ}{g}_{ik}$ längs $\overset{\circ}{X}^i$. Der geometrische Sinn der allgemeineren Bedingung (5) besteht in der konformen Änderung des metrischen Tensors g_{ik} längs X^i ***:

$$X^\alpha \frac{\partial}{\partial x^\alpha} g_{ik} = 2\rho\, g_{ik}\,. \tag{7}$$

II. In dieser Note bleiben wir bei derselben Ansicht, wie in der Theorie des zylindrischen $\overset{\circ}{R}_5$, dass nämlich keine von den zu X^i orthogonalen nichtholonomen Hyperflächen als vierdimensionale Raum-Zeit-Welt sich von den andern auszeichnet, sondern, dass wir alle solche Hyperflächen gemeinsam unmittelbar wahrnehmen. Dann ist es aber nicht mehr möglich die Metrik des R_5 mit der Raum-Zeit-Weltmetrik zu identifizieren; vielmehr können dann nur die Verhältnisse der fünfdimensionalen g_{ik} (z. B. die $\frac{g_{ik}}{g_{55}}$) einen unmittelbaren physikalischen Sinn haben. Wenn man von den dx^i ($i = 1, \ldots 5$) die ersten vier der Einfachkeit halben den Differentialen der raum-zeitlichen Koordinaten gleichsetzt, to sieht man sofort, dass die Längen im R_5 werden in Einheiten gemessen, die sich von der raum-zeitlichen Einheit (1 cm) durch einen Faktor, der eine im allgemeinen willkürliche (stetige) Funktion der $x^1 \ldots x^5$ sein kann, unterscheiden. Wir schreiben also:

$$ds = \sigma\, \overset{\circ}{ds};\; g_{ik} = \sigma^2\, \overset{\circ}{g}_{ik}, \tag{8}$$

wobei $\overset{\circ}{ds}$ in Zentimetern gemessen wird.

Auch andere Tensoren unseren R_5, die eine physikalische Deutung haben, müssen sich ebenfalls in der Richtung X^i konform ändern.

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* $\varkappa = \frac{4K}{c^4}$, K — Newtonsche Gravitationskonstante.

** Der Strich bedeutet die Projektion des überstrichenen Tensors orthogonal zu X^i.

*** Vgl. H. Mandel, ZS. f. Phys., 54, 564 (1929).

Mit $\sigma(x^1 \ldots x^5) = 1$ definiert man die spezielle Hyperfläche im R_5, für die die Längeneinheit mit der gewöhnlichen (1 cm) zusammenfällt. (In der früheren Theorie des zylindrischen $\overset{\circ}{R}_5$ hatte man $\sigma \equiv 1$.) Man sieht, dass der 5-dimensionale Raum der Bornschen Elektrodynamik besitzt keinen so einfachanschaulichen Sinn, wie der der Maxwellschen Elektrodynamik.

Bei diesem Vergleich zogen wir nur solche fünfdimensionale Längen in Betracht, deren Richtungen orthogonal zu X^i, d. h. vollständig in der vierdimensionalen Raum-Zeit-Welt lagen. Für Längen, die die Richtung von X^i haben ($ds^i = X^i\, ds$), liegt überhaupt keine Möglichkeit vor sie unmittelbar mit den raum-zeitlichen Längen zu vergleichen. Die Wahl der Längeneinheit für diese Richtung könnte allerdings nur irgendwie physikalisch begründet werden, und vorläufig bleibt der Raum R_5 in dieser Hinsicht noch vollkommen unbestimmt.

III. An dieser Stelle merken wir, dass solcheine Unbestimmtheit auch in der Theorie des zylindrischen $\overset{\circ}{R}_5$ vorliegen muss. Wenn man nämlich $\overset{\circ}{R}_5$ in der Richtung $\overset{\circ}{X}^i$ a-fach zusammenzieht

$$\overset{\circ}{X}'^i = a\overset{\circ}{X}^i,\quad \overset{\circ}{X}_i' = a^{-1}\overset{\circ}{X}_i, \tag{9}$$

so muss dies tatsächlich keine Änderung in den vierdimensionalen physikalischen Gesetzen hervorrufen. Die Gesetze der fünfdimensionalen Feldtheorie müssen folglich die Invarianzeigenschaft in bezug auf die Transformationen (9) («a-Transformationen») besitzen. In unseren früheren (und anderen) Arbeiten wurde auf eine derartige Invarianzeigenschaft nicht geachtet, was sich darin äusserte, dass man den Vektor, der zur Bildung der Viererprojektionen der Fünfertensoren dient, als Einheitsvektor im $\overset{\circ}{R}_5$, d. h. seine Länge gleich 1 cm wählte. Die Inkonsequenz dieser Annahme folgt unmittelbar aus der Idee der Invarianz gegenüber a-Transformationen, d. h. aus der Willkür bei der Wahl der Längeneinheit längs $\overset{\circ}{X}^i$ im $\overset{\circ}{R}_5$.

Mit (9) wird $\overset{\circ}{R}_5$ in solch einen $\overset{\circ}{R}'_5$ übergeführt, wo der Projektionsvektor $V'^i \equiv \overset{\circ}{X}^i = a^{-1}\overset{\circ}{X}'^i$ ($\overset{\circ}{X}'^i$ — Einheitsvektor im $\overset{\circ}{R}_5'$) die Länge a^{-1} cm hat. Bei dieser Transformation (9) ändert sich allerdings der Neigungswinkel $\overset{\circ}{\Theta}$

$$\operatorname{cth}\overset{\circ}{\Theta} = \frac{Y_\alpha X^\alpha}{|\overline{Y}|} = \frac{e}{mc^2\sqrt{\varkappa}}\ *,$$

den die geodätische Linie — die fünfdimensionale Weltlinie eines Massenpunktes (mit Ladung e und Masse m) — mit $\overset{\circ}{X}^i$ bildet. Im $\overset{\circ}{R}'_5$ bekommen wir:

$$\operatorname{cth}\Theta' = \frac{Y_\alpha \overset{\circ}{X}'^\alpha}{|\overline{Y}|} = a\frac{Y_\alpha \overset{\circ}{X}^\alpha}{|\overline{Y}|} = a\operatorname{cth}\overset{\circ}{\Theta} = \frac{ae}{mc^2\sqrt{\varkappa}}. \tag{10}$$

Die Bewegungsgleichungen, d. h. Gleichungen der geodätischen Linie im $\overset{\circ}{R}_5$, können nämlich Gleichungen der geodätischen Linie in jedem solchen $\overset{\circ}{R}'_5$ ebenfalls darstellen, wenn man dort den Geschwindigkeitsvektor durch Gleichsetzen der unteren Komponenten Y_i ($i = 1, \ldots, 5$) im $\overset{\circ}{R}'_5$

* Vgl. z. B. unsere Darstellung der fünfdimensionalen Feldtheorie in «Annals of the University Leningrad», Ser. of phys. sc., 2, 85 (1936). Wir bedienen uns hier der Bezeichnungen jenes Artikels.

definiert. Im üblichen speziellen «Z-Koordinatensystem» (in dem $\overset{\circ}{X}^i = 0$, $i = 1, \ldots, 4$), erhält man dann, dass im $\overset{\circ}{R}_5$ und im $\overset{\circ}{R}'_5$ auch obere Komponenten Y^i ($i = 1, \ldots, 4$) einander gleich, und nur Y^5 verschieden sind. Dasselbe bezieht sich auch zur Wellengleichung, da die Komponenten ∇_i ($i = 1, \ldots, 5$) müssen im $\overset{\circ}{R}_5$ und im $\overset{\circ}{R}'_5$ dieselben sein.

Von allen solchen $\overset{\circ}{R}'_5$ wählen wir nun dasjenige $\overset{\circ}{R}^{\mathrm{I}}_5$, in welchem die Weltlinie des Elektrons eine Nullinie und $\operatorname{cth} \Theta^{\mathrm{I}}$ dieser Weltlinie gleich Eins ist *. Mit dieser naheliegenden (eventuel einer geometrischen und nicht physikalischen) Forderung wird der Wert $a_{\mathrm{I}} - \frac{m_0 c^2 \sqrt{\varkappa}}{e_0}$ ausgezeichnet (e_0, m_0 — Ladung und Masse des Elektrons), und kann die Längeneinheit längs $\overset{\circ}{X}^i$ definiert werden **. Daraus folgt, dass die Länge des Projektionsvektors $V^i = \overset{\circ}{X}^i = a_{\mathrm{I}}^{-1} \overset{\circ}{X}^{\mathrm{I}i}$ gleich a_{I}^{-1} cm ist und eine natürliche Längeneinheit darstellt. [Im uhrsprünglichen $\overset{\circ}{R}_5$ wurde diese Längeneinheit willkürlicherweise gleich 1 cm gesetzt; wir können nun sagen, das dort (im $\overset{\circ}{R}_5$) die Längen der Richtung $\overset{\circ}{X}^i$ in solchen natürlichen Längeneinheiten gemessen wurden.]

IV. Nach diesen vorläufigen geometrischen Ausführungen kehren wir zum Tensor F_{ik} der Bornschen Theorie zurück. Die Untersuchung dieses Tensors führt uns zu weiteren Folgerungen bezüglich die Eigenschaften des Raumes R_5 der Bornschen Theorie.

Wir führen in (6) nach (8) $g_{ik} = \sigma^2 \overset{\circ}{g}_{ik}$ ein. Da die Komponenten des Vektors X_i des R_5 bei $\sigma = 1$ in die Komponenten des Vektors $\overset{\circ}{X}_5$ des $\overset{\circ}{R}_5$ übergehen müssen, setzen wir $X_i = \sigma \overset{\circ}{X}_i = \sigma \sqrt{\varkappa} \Phi_i$ ($\Phi_i, i = 1, \ldots, 4$ — Viererpotential). Es folgt dann aus (6):

$$2 \overline{\nabla_i X_k} = \sigma \sqrt{\varkappa} \left(\frac{2}{\sqrt{\varkappa}} \rho \sigma \overset{\circ}{g}_{ik} + \mathfrak{M}_{ik} \right). \tag{11}$$

Der Vergleich mit F_{ik} ergibt: $\frac{2}{\sqrt{\varkappa}} \rho \sigma = \frac{b}{2} = \text{const}$.

Daraus folgt:

$$\rho \sigma = X^a \frac{\partial}{\partial x^a} \sigma = \frac{d\sigma}{dl} = \frac{\sqrt{\varkappa}}{\sqrt{8}} b; \quad \sigma = \frac{\sqrt{\varkappa}}{\sqrt{8}} bl + \sigma_1. \tag{12}$$

Beim Abwesen des elektromagnesischen Feldes ($\mathfrak{M}_{ik} = 0$) würden zu X_i orthogonale Hyperflächen existieren; der Tensor $\overline{\nabla_i X_k}$ würde dann den zweiten Fundamentaltensor und ρ die (konstante) skalare Gaussche Krümmung dieser Hyperflächen darstellen. Im unseren Fall können wir ρ als Gaussche

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* Die Wellengleichung des Elektrons hat dann die Form:

$$\sum_{\mu=1}^{4} \alpha_\mu \overset{\circ}{X}^{i\nu} \nabla_\nu \psi = 0.$$

** Im zylindrischen Raume ist eigentlich diese Wahl der Längeneinheit ebenso willkürlich, wie die frühere (des Raumes $\overset{\circ}{R}_5$). Ihr Vorteil zeigt sich aber bei der weiter folgenden kosmologischen Betrachtung im R_5 der Bornschen Theorie, da dort, wie wir sehen werden, die Invarianz gegenüber «a-Transformationen» nicht mehr bestehen bleibt.

Krümmung der nichtholonomen zu X^i orthogonalen Hyperflächen bezeichnen. Aus (12) ersieht man, dass sie sich längs X^i reziprok zu σ ändert.

Wenn man in (12) σ_1 als klein vernachlässigen kann*, so sind unsere Hyperflächen einfach konzentrische vierdimensionale sphärische Hyperflächen mit dem Radius l. Von ihnen wird durch $\sigma = 1$ diejenige ausgezeichnet, deren Krümmung $\rho_{\sigma=1}$ der Krümmung der Raum-Zeit-Welt gleich sein muss. Diese Rechnung wurde aber in dem Raume R_5 durchgeführt, wo der Einheitsvektor X^i (wegen $g_{\alpha\beta} X^\alpha X^\beta = g^{\alpha\beta} X_\alpha X_\beta = \overset{\circ}{g}{}^{\alpha\beta} \overset{\circ}{X}_\alpha \overset{\circ}{X}{}^\alpha$) nach unserer obigen Voraussetzung die natürliche Länge gleich a_I^{-1} cm hat. Also sind die Längen $\frac{1}{\rho_\sigma}$ und l ebenfalls nicht in Zentimetern, sondern in unseren natürlichen Einheiten gemessen. Nun geht man vom R_5 mittels der a-Transformation (Zusammenziehung längs X^i a-fach) zum R_5^I über (man setzt: $a = a_I$). Im R_5^I sind nach unserer Behauptung die Längen der Richtung X^i bereits in Zentimetern gemessen. In dem wir nun im R_5^I den «echten» Einheitsvektor X^{Ii} einführen, berechnen wir die Krümmung ρ_5^I im R_5^I nach der Formel: $\rho^I \sigma = X^{I\alpha} \frac{\partial}{\partial x^\alpha} \sigma = \frac{d\sigma}{dl^I}$. Wir erhalten somit den Krümmungsradius in Zentimetern gemessen:

$$\frac{1}{\rho^I_{\sigma=1}} = l^I_{\sigma=1} = \frac{1}{a_I \rho_{\sigma=1}} = \frac{l}{a_I} = \frac{\sqrt{8}}{a_I \sqrt{\varkappa}\, b} = \frac{\sqrt{8}\, e_0}{\varkappa m_0 c^2 b^*}. \tag{13}$$

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* Wir beachten, dass $\mathfrak{M}_{ik} = 0$ eine notwendige Bedingung für $\sigma_1 = \text{const}$ darstellt.

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PHYSIQUE

LES SPECTRES DE COMBINAISON DE LA SOLUTION DE L'EAU DANS LE DIOXANE ET LA PYRIDINE

Par G. S. LANDSBERG, Membre-Correspondant de l'Académie, et V. S. MALYSCHEV

Les études de Landsberg et Ukholine(1) sur la variation de la fréquence propre des oscillations du groupe hydroxyle dans l'eau et dans l'alcool méthylique en fonction de la densité démontrent clairement que la raie nette de diffusion de combinaison, caractéristique pour le groupe OH, correspond seulement aux molécules isolées. Dans le cas où la distance entre les molécules est suffisamment petite, la fréquence des oscillations est perturbée par l'action des molécules voisines et la raie se transforme en une bande. Apparemment, l'action perturbante principale vient de l'atome d'oxygène de la molécule voisine(2).

Dans les études mentionnées, l'augmentation de la distance entre les molécules et l'isolation de celles-ci étaient obtenues par la diminution de la densité par échauffement et par la transition de l'état liquide à l'état gazeux. Il reste incertain si la cause de la perturbation réside dans l'existence des moments électriques dans les molécules inter-agissantes (le moment d'une molécule d'eau est $1.8 \cdot 10^{-18}$, celui d'une molécule d'alcool méthylique $1.7 \cdot 10^{-18}$), ou si le rôle principal est joué par l'interaction de l'hydrogène avec les oxygènes de deux molécules voisines (liaison hydrogénique), ou, enfin, si l'effet essentiel est l'interaction de deux groupes hydroxyles (liaison hydroxylique)(3).

C'est pourquoi il est intéressant d'augmenter la distance entre les molécules contenant des groupes hydroxyles en les dissolvant dans des milieux convenablement choisis. On peut varier alors la nature du dissolvant, de façon à étudier le caractère de l'interaction supposée. Par exemple si l'on dilue l'eau par du dioxane ($C_4H_8O_2$), l'on crée un entourage de molécules à moment électrique résultant réduit ($\mu < 0.4 \cdot 10^{-18}$), mais contenant des oxygènes. Si l'on substitue au dioxane la pyridine, l'on introduit des molécules à moment électrique sensible ($2.3 \cdot 10^{-18}$), mais privées d'oxygène. L'azote composant la pyridine est caractérisé par une liaison avec l'hydrogène plus lâche que celle de l'oxygène.

Le présent travail expose les résultats de l'étude du spectre de combinaison de l'eau diluée par du dioxane ou de la pyridine.

Il est possible d'obtenir des échantillons de dioxane et de pyridine ne fluorescent que très faiblement, si on les purifie très soigneusement et les soumet ensuite à une distillation répétée, en terminant par leur distillation dans le vide, d'usage dans les expériences de diffusion de la lumi-

ère. L'étude était effectuée dans un dispositif du type de Wood, réalisé à l'aide d'appareils spéciaux de la firme Zeiss. Le spectrographe était de Zeiss à trois prismes, muni d'un caméra court à grande luminosité (f=12 cm).

Le spectre de l'eau purifiée obtenu en 3 minutes sur des plaques à grande sensibilité exhibe la bande large ordinaire, à maximum principal 3460 cm^{-1}. Les variations que l'on observe lorsqu'on dilue l'eau par du dioxane sont montrées dans le tableau. Quelques cas caractéristiques sont illustrés par des microphotogrammes (voir la fig. 1).

	Largeur de la raie *	Position du maximum
Eau pure	420 cm^{-1}	3 460 cm^{-1}
Dioxane avec 20% d'eau	220 cm^{-1}	3 520 cm^{-1}
» » 10% »	200 cm^{-1}	3 520 cm^{-1}
» » 5% »	160 cm^{-1}	3 510 cm^{-1} — 3 590 cm^{-1}
» » 3½% »	146 cm^{-1}	3 510 cm^{-1} — 3 600 cm^{-1}
» » 2% »	135 cm^{-1}	3 520 cm^{-1} — 3 600 cm^{-1}
» » 1% »	135 cm^{-1}	3 520 cm^{-1} — 3 600 cm^{-1}
Dioxane pur	—	3 512 cm^{-1} — 3 596 cm^{-1}

La position des maxima ne pouvant pas être déterminée très exactement, les chiffres sont arrondis. A partir de la concentration de 5%, une séparation en deux maxima s'ébranche, le maximum à ondes plus longues (3510 cm^{-1}) étant considérablement plus intense. A mesure que l'on dilue, les maxima se déplacent légèrement dans le sens des grandes fréquences

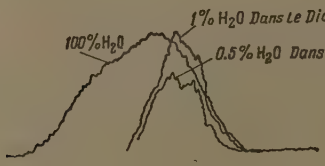

Fig. 1.

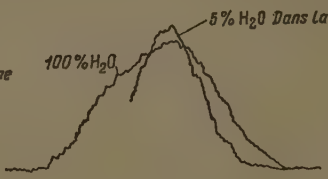

Fig. 2.

et leurs intensités s'égalisent graduellement. A des concentrations d'eau plus petites (2%, 1%) la séparation des maxima est plus nette et leur position peut être déterminée de façon plus exacte. Les deux maxima qui caractérisent le «dioxane pur» sont dus à des vestiges d'eau qu'il est très difficile d'extraire du dioxane. Dans notre cas, le contenu d'eau dans le dioxane pur était vraisemblablement d'environ ¼—½%.

Les maxima observés paraissent quelque peu plus larges qu'il ne le faudrait, car le groupe de raies existant (3650 Å, 3655 Å et 3663Å) est un groupe que laisse passer un filtre noir et la dispersion linéaire du spectrographe est petite (47 Å/mm).

D'autre part, l'impossibilité de distinguer dans ces maxima une structure correspondant à la complexité du groupe existant démontre que les bandes excitées par chacune des trois raies se superposent, chaque bande n'étant pas plus étroite que 20^{-1}—30^{-1}cm.

Par conséquent, l'expérience démontre que la bande large de l'eau se décompose en deux bandes relativement étroites lorsque l'eau est fortement diluée dans du dioxane. L'interprétation de cette paire est assez difficile.

* Evaluée d'après les données du comparateur.

Il est possible qu'elle représente les vestiges de la bande de l'eau rétrécie et de la raie qui caractérise les molécules d'eau isolées (cette raie étant déformée par l'action du dioxane), comme dans le cas de vapeur d'eau à 135 atm.(1).

Néanmoins une interprétation pareille, fondée sur certaines conceptions du mécanisme de la dissolution, devrait être sujette à des vérifications expérimentales avec d'autres dissolvants.

On note aussi une certaine similitude entre les doublets obtenus et les doublets observés par Hibben (4) dans l'eau de cristallisation, où une molécule d'eau est pareillement entamée par un nombre limité de voisins aqueux. On peut en tous cas constater que la dilution de l'eau par le dioxane conduit en général aux mêmes effets que la diminution de la densité de l'eau par échauffement: à un fort rétrécissement de la bande et à son déplacement vers le domaine des fréquences élevées, avec une tendance à se transformer en raie nette qui caractérise les molécules isolées.

Le fait que l'aspect du spectre ne varie que peu lorsque l'on passe d'une dilution de 20% à 10% et même à 5% et varie fortement quand on passe à des dilutions plus fortes se trouve probablement en rapport avec le fait que dans les trois premiers cas on a respectivement 1, 2 et 4 molécules de dioxane pour chaque molécule d'eau; dans le cas d'une dilution de 2% on a dix molécules de dioxane pour chaque molécule d'eau. Par conséquent, les conditions favorables à l'isolation mutuelle des molécules de l'eau ne sont réalisées que dans le cas de fortes dilutions.

Nos résultats sont confirmés par les observations de Gordy (5) sur le spectre infra-rouge de l'eau et de ses solutions dans le dioxane. Gordy observe une substitution graduelle du maximum 2.8 μ (proche du maximum de la vapeur d'eau) au maximum d'absorption infra-rouge 3 μ (caractéristique pour l'eau liquide). En contradiction avec nos résultats, Gordy ne trouve pas de différence entre les spectres d'absorption à 4% d'eau et ceux à 2% et à 0.5%. Ceci peut apparemment être expliqué par ce que la spectroscopie d'absorption infra-rouge n'est ordinairement pas en état de découvrir des détails, que décèle la méthode de la spectroscopie de la lumière diffusée.

Les résultats obtenus avec la pyridine peuvent être résumés de façon suivante: la position du maximum de la bande de l'eau reste invariable à toutes les dilutions et coïncide avec le maximum de l'eau pure [3460 cm^{-1}, voir la fig. 2]. A mesure que le contenu d'eau diminue, la bande devient néanmoins beaucoup plus étroite. L'étude des petites concentrations est rendue difficile par le fait qu'on observe dans la pyridine pure une raie faible, située approximativement à la place du maximum de l'eau (3460 cm^{-1}). La variation de l'intensité de cette raie, lorsque l'on ajoute de l'eau, exclut la supposition que cette raie peut être due à des vestiges d'eau (dans la «pyridine pure»).

Nonobstant cette complication, l'on peut vraisemblablement affirmer, que la dissolution dans la pyridine ne permet pas de réaliser des conditions favorables à «l'isolation» des molécules d'eau. Des études ultérieures doivent élucider le rôle que joue le moment électrique du dissolvant.

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PHYSICS

LONG PERIOD RADIOACTIVITY IN SILVER, CAESIUM AND INDIUM ACTIVATED BY SLOW NEUTRON BOMBARDMENT

By K. ALEXEEVA

(Communicated by S. I. Wawilow, Member of the Academy, 4. II. 1938)

In a previous work (1) the author has revealed and investigated artificial radioactivity in antimony, due to slow neutrons and exhibiting a 45 day half-life. In the present work to establish long period artificial radioactivity for other elements, the author has subjected the following substances to prolonged bombardment lasting 6—7 months with slow neutrons: Be, C, NaCl, Mg, Al, Ti, V, Cr, Mn, Fe, Co, Ni, Cu, Zn, Ga, Se, Y_2O_3, Mo, Ru, Rh, Pd, Ag, Cd, In, Sn, Sb, Te, CsCl, $Pr(NO_3)_3$, Ta, W, Re, Ir, Pt, Au, Tl, Bi.

Previous to bombardment the samples were tested for absence of activities from radioactive impurities. As before, bombardment was made by photo-neutrons, stopped in paraffine, created under the action on berillium (600g) of γ-rays from a Ra+MsTh-preparation (equivalent to 650 mg of radium with respect to γ-rays). The activity of the samples was measured by means of a Geiger-Müller counter (walls, 0.5 cm thick; diameter, 2 cm; length, 8 cm), supplied with a window of 1.5 x 7 cm^2, shut with a thin copper foil 13 μ thick.

An analysis of the respective induced activities of the said substances made after bombardment gave, in addition to the long periods already known for Co(2), Sb (1, 3), Ta (4) and Ir (5), long periods for s i l v e r, c a e s i u m and i n d i u m.

S i l v e r. Four samples were subjected to bombardment: a) electrolitic sheet silver, 0.45 mm thick; b) sheet Kahlbaum silver, 0.35 mm thick; c) Ag_2O(8.7g), Kahlbaum; and d) $AgNO_3$(36g), Goslaborreactive. The surface area of the samples was about the same as that of the counter window. Sample a) was exposed to bombardment for six months; the control-samples b), c) and d), for 2 months each. All four samples exhibited an increase of activity nearly proportional to the silver content and the length of exposure.

The decay curve of silver [sample a), long period] is given in Fig. 1. During the 80 days following bombardment, a rather negligeable decrease in activity took place (almost within the limits of measurement error). From the smooth logarithmical curve (Fig. 2), the half-life of silver is assigned a value of the order 1.5—2 years. The half-absorbing layer for electrons (positive or negative) has been estimated approximately as 0.15 mm Al. γ-rays have been observed too.

Previous to bombardment sample a) was spectroscopically examined to determine the degree of its purity. Slight admixtures of the following substan-

ces have been found: 1) Si, Pb, Sn, Cu and 2) Al, Ca, Mg, Fe, Bi [a rough qualitative estimation of their content in sample a) gave approximate values of the orders 0.1—0.01% and 0.01—0.001% respectively]. Since neither Si, Pb, Ca, nor As or S, frequently met with as admixtures to silver, had been observed under long bombardment, a supplementary control-experiment was performed, envolving their bombardment with slow neutrons under analogous conditions. After one month's exposure none of them did exhibit an increase of activity, similar to that observed in silver.

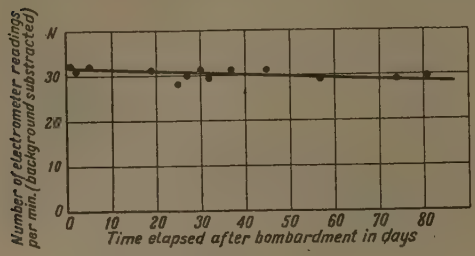

Fig. 1.

The reaction most probable to occur in the course of bombardment of heavy elements with slow neutrons is that of neutron-capture. It should, therefore, be assumed that the long period assigned to silver belongs to its isotope Ag^{108} or Ag^{110}. From experimental data of different authors available nowadays, the radioactive isotope Ag^{108} is assigned a short period of 2.3 min.; the radioactive isotope Ag^{110}, a period of 22 sec. Thus it seems most natural to assume that the radioactive isotope exhibiting the long period observed, is an isomere of one of the radioactive isotopes cited above. In this case the long period should correspond to the initial state of the nucleus; the short one, to its excited state.

However, the ratio of the isomere periods proves to be too large in both cases and considerably exceeds the respective ratios for all isomeres known till now: Br^{80}(6), Ag^{106}(7), Sr^{89}(8), In^{114} and In^{116}(9). Therefore it may after all be possible that Ag^{108} (2.3 min.)and Ag (22 sec.) are the isomeres in question. This period has, however, not been experimentally established neither with γ-rays, nor with fast neutrons. It should be noticed in this connection that lately data were reported(10) on the variability of the ratio of isomere activities, according to the way in which they are created.

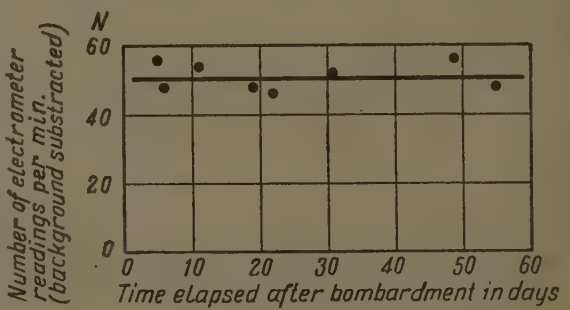

Fig. 2.

Caesium. Caesium was subjected to bombardment under the form of a CsCl compound of Kahlbaum (2g). A parallel bombardment of a crystal of NaCl made under similar conditions, showed that the long period obtained did not belong to chlorine.

The decay curve is given in Fig. 2. From this curve it may be seen that the half-life of caesium should not be less than 1 year.

Spectral analysis produced previous to bombardment revealed the presence of the following admixtures: Ca, Ti, Na, Rb, as well as of Al, Mg, Cu, in quantities of the orders 0.01% and 0.001% respectively. All of them, except Rb, have been examined, and exhibited no long period activities under the given conditions.

Indium. Schuchhardt's metallic indium was taken for examination (0.3 g, surface area ~ 0.5 cm^2). After a six months' bombardment and a short period disintegration, the existence of a long period activity was established. The decay curve is given in Fig. 3. From this curve, the long half-life of indium may be assigned the value of 60 ± 10 days.

Previous to bombardment the sample was subjected to spectral analysis which revealed the presence of small quantities of Mg, Pb, Cu, Ag and Ti (0.01—0.001%), which, under given experimental conditions, produced no activities with the long period cited above.

Lawson and Cork [9] have studied induced radioactivity in indium bombarded with slow neutrons, fast neutrons (14—20 MeV), and deutons. Bombarding indium with fast neutrons, they succeeded in obtaining periods of 13 sec., 72 sec., 54 min., 4.1 hours and 50 days. The 4.1-hour and 50-day periods have been assigned by them to the radioactive isotope In^{114}. Using slow neutrons and deutons they have obtained periods of 13 sec., 54 min.

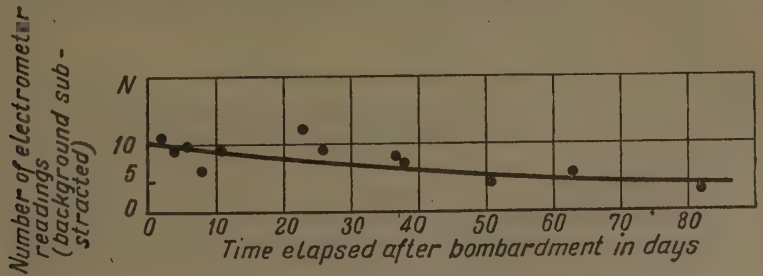

Fig. 3.

and 4.1 hours; the 50-day period has not been established in this case. The 60 ± 10-day period which I found for indium bombarded with slow neutrons, is, evidently, identical to the 50-day period, reported by the authors cited above for indium bombarded with fast neutrons.

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PHYSICS

ABSORPTION SPECTRA OF PRASEODYMIUM AND NEODYMIUM IN HEAVY WATER

By S. M. KATCHENKOV

(Communicated by S. I. Wawilow, Member of the Academy, 8. II. 1938)

The absoprtion spectra of water solutions of salts of rare earths' elements have been closely examined by Prandtl and Scheiner(1). These authors showed that the spectra consist of a series of comparatively narrow bands in the visible and the ultra-violet regions. The presence of narrow bands in the absorption spectra of rare earths' elements, in contrast with solutions of other elements, showing wide bands, may be explained by the peculiar structure of their atoms. The inner film of the latter $4f$ is in the state of being gradually filled by electrons with an increase of the atom number, whereas the other and even the upper films $5s$, $5p$, $5d$ and $6s$ have the same complete structure as in all rare-earths' elements.

The absorption spectra are caused by the initial and final quantum states which arise at the absorption of light from the different orientations of the spin and of the orbital moments of electrons inside the film $4f$.

There is no theoretical interpretation of the absorption spectra of rare earths' elements till now, and the principles of the absorption have not yet been established. The object of the present work was to study the absorption spectra of the praseodymium and neodymium sulphate solutions in heavy water. Kahlbaum's salts $Pr_2(SO_4)_3$ and $Nd_2(SO_4)_3$ were used for the solutions. They were previously warmed to 400—500°C to make them free of the molecules of ordinary water. There were two kinds of heavy water for the solutions: one of 93% (from Professor A. I. Brodsky's Laboratory at Dnepropetrovsk), and the other of 100% (from the Norwegian firm Norsk Hydro-Electrisk Kvealstof A/S at Rjukan).

For comparison, ordinary water was taken, after having been repeatedly distilled in a platinum flask at the Laboratory of Standards of the All-Union Scientific Research Institute of Metrology in Leningrad. The concentration of the solution both in ordinary and in heavy water was of 5%. For measuring the absorption of the solutions two identical quartz cuvettes of 1 cm^3 capacity and 20 mm long, supplied with plane-parallel windows, were made. The photos were taken in quartz spectrographs: of Zeiss with a dispersion of 150 Å/mm in the region of 5000 Å, and of Hilger with the dispersion of 70Å/mm in the region of 5000 Å. A 50-Watt incandescent lamp, fed by direct current, served as a source of light. The photos with the solutions of the corresponding salt in ordinary and in heavy water were taken under the same conditions, side by side, without moving the box, on plates NIKFI with a sensitivity of 500—700 H and D.

The very first prints of the spectra obtained from a solution of neodymium sulphate in 93% heavy water showed a shift of the absorption spectrum towards the violet side as compared to the spectrum of the ordinary water solution. With a heavy water of 100% these displacements were still more marked. The absorption spectrum of the praseodymium sulphate solutions was photographed under the same conditions. A shift of the absorption spectrum of praseodymium forward the violet region was observed in heavy water of 100% as well as in ordinary water.

The photos were taken at room temperature. Yet a dependence of the absorption spectrum on temperature was observed, viz., a change in temperature was followed by a shifting of absorption bands which was not identical in heavy and ordinary water solutions. The displacement of the absorption spectra of solutions of praseodymium and neodymium sulphates were measured with a Zeiss comparator and proved to be of the order of 30—35 cm^{-1}

The absorption spectrum of neodymium in 100% heavy water (in the borders) and in ordinary water (in the middle) in the region of 4550—4780 and 4980—5340 Å

for both elements. It is worthy noting too that another effect of heavy water on the absorption spectrum consists in the reduced intensity of the latter, and in widening of the bands by 2—5%. The dissolution of the sulphates of rare earths' elements (equal concentrations) in heavy water proceeds slower and gives more intensely coloured solutions than in ordinary water.

Summary

1. The influence of water molecules on the inner electrons of rare earths' elements of prasedymium amd neodymium is established.

2. The molecules of ordinary (H_2O) and of heavy water (D_2O) have a different influence on the absorption spectrum. In heavy water the absorption spectrum shifts towards the short wave zone and produces a slight change in the structure of bands.

3. Different physical and chemical properties of heavy water and ordinary water account for an unequal change in the absorption spectrum due to temperature.

4. The dissolution of neodymium sulfates in heavy water proceeds slower and gives a brighter colour than in ordinary water.

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PHYSIQUE

L'INTENSITÉ DES RAIES SPECTRALES DANS UN ARC À ÉLECTRODES DE CHARBON

Par S. L. MANDELSTAM

(Présenté par S. I. Wawilow, de l'Académie, le 9.II.1938)

1. Un arc brûlant librement dans l'air est l'une des sources d'excitation du spectre les plus communément employées dans l'analyse spectroscopique. En particulier, on emploie d'ordinaire pour l'analyse qualitative un arc à électrodes de charbon, en introduisant la matière étudiée dans une cavité pratiquée dans l'une des électrodes. Lorsque de tels arcs, comme en général toute source, sont employés dans les applications pratiques de l'analyse spectrale, surgit une question d'importance primordiale concernant l'influence d'éléments supplémentaires renfermés dans l'échantillon sur l'intensité absolue et relative des raies de l'élément étudié et de l'élément fondamental.

Webb [1] et d'autres auteurs [2] ont récemment observé un cas caractéristique d'une telle influence, qui paraît à première vue paradoxale: l'introduction d'une grande quantité de NaCl dans l'arc entre les électrodes de charbon provoque un renforcement considérable des raies d'une série d'éléments: Ca, Ti, V, Fe etc., contenus dans les électrodes sous forme de légères impuretés. Simultanément avec Webb cet effet a été observé par S. Borovik [3], qui a trouvé que le KCl et quelques autres sels agissent pareillement et que les raies d'arc des additions sont renforcées, alors que les raies d'étincelle sont affaiblies. Willey [4] suppose que cet effet peut être expliqué par la stimulation de la radiation par les chocs de deuxième espèce. Cette explication nous semble néanmoins très peu probable, car il est difficile de concevoir que les niveaux énergétiques coïncident d'une façon assez exacte pour un nombre si grand de combinaisons entre les éléments. A notre avis il est plus vraisemblable que ce phénomène est dû à une variation de la température de l'arc.

2. On peut considérer comme établi que dans la colonne de l'arc à pression atmosphérique un équilibre thermodynamique approximatif a lieu entre les électrons et les molécules du gaz [5]. Par conséquent, l'ionisation et l'excitation ont en essentiel un caractère thermique. Le degré d'ionisation dans ce cas peut être calculé par la formule de Saha:

$$\frac{x^2}{1-x^2}\,p = \frac{(2\pi m)^{\frac{3}{2}}\,(kT)^{\frac{5}{2}}}{h^3}\,e^{-\frac{V_i}{kT}} \qquad (1)$$

et la distribution des atomes par niveaux d'énergie peut être trouvée par la formule de Boltzmann.

L'introduction dans l'arc de quantités considérables de matière à potentiel d'ionisation bas facilite l'apparition du nombre d'électrons indispensable à la conservation du courant initial et par suite la température de l'arc et le gradient du potentiel décroissent. Par exemple, dans un arc brûlant librement $T \approx 7000°$ (6), alors que dans un arc brûlant dans la vapeur de Na $T \approx 3000°$ (8). On peut poser en première approximation la température de l'arc proportionnelle au potentiel d'ionisation du gaz dans lequel il est plongé*.

3. Si certains éléments supplémentaires en plus du carbone sont contenus ou introduits dans les électrodes de l'arc, alors leur degré d'ionisation est donné par la formule:

$$\frac{x}{1-x}\,\frac{x_{res}}{1-x_{res}}\cdot p = \frac{(2\pi m)^{\frac{3}{2}}(kT)^{\frac{5}{2}}}{h^3}\,e^{-\frac{V_i}{kT}}, \qquad (2)$$

x_{res} étant le degré d'ionisation de tout le gaz. Si la quantité d'additions est très petite, le potentiel d'ionisation est déterminé en essentiel par le carbone et ses composés. On peut dans ce cas poser pour un arc à courant 5—6 ampères et à intervalle 5—6 mm: $x_{res} \approx 1 \cdot 10^{-3}$ (7), $T \approx 7000°$.

Dans ces conditions les éléments à potentiels d'ionisation bas se trouvent presque entièrement ionisés. Le tableau renferme le rapport du nombre d'atomes ionisés et du nombre d'atomes neutres, calculé par la formule (2):

$$\frac{N_i}{N_0} = \gamma^\circ = \frac{x}{1-x}.$$

Élément	V_i	$T = 7000°$	$T = 3000°$
Ca	6.1	55	$8 \cdot 10^{-6}$
Ti	6.8	17	$5 \cdot 10^{-7}$
Cu	7.7	3.6	$2 \cdot 10^{-8}$
B	8.3	1.3	$2 \cdot 10^{-9}$

La dernière colonne renferme les valeurs de γ° pour $T = 3000°$, c'est le cas quand, par exemple, de grandes quantités de NaCl ont été introduites, si l'on suppose en accord avec les données de Hoermann(7) que x_{res} reste invariable. En réalité, à ce qu'il semble, de nombreux éléments, comme par exemple Ca, Ti, Mg etc., ont à $T = 7000°$ un degré d'ionisation encore plus élevé par suite de l'existence de niveaux métastables; on peut apparemment prendre pour le Ca $V_i \leqslant 4.5$ V dans la formule (2) (10).

L'intensité des raies de résonance des atomes neutres I_r et des atomes ionisés I'_r est donnée par les formules:

$$\left.\begin{aligned} I_r &= cN_0 e^{-\frac{V_r}{kT}} = c(1-x)Ne^{-\frac{V_r}{kT}} = c\,\frac{N}{1+\gamma^\circ}\,e^{-\frac{V_r}{kT}}, \\ I'_r &= c'N_i e^{-\frac{V'_r}{kT}} = c'xNe^{-\frac{V'_r}{kT}} = c'\,\frac{N\gamma^\circ}{1+\gamma^\circ}\,e^{-\frac{V'_r}{kT}} \end{aligned}\right\} \qquad (3)$$

où $N = N_1 + N_0$.

* Engel et Steinbeck (9) donnent $T = \frac{2lV_i}{29\,k}$ pour de grands courants.

Les calculs montrent que lorsque la température croît de 3000 à 7000°, I_r augmente d'abord par suite de la croissance du facteur $e^{-\frac{V_r}{kT}}$; à mesure que l'ionisation croît ensuite, la décroissance du facteur $\frac{1}{1+\gamma}$ devient si rapide qu'elle surpasse l'action croissante du facteur $e^{-\frac{V_r}{kT}}$; I_r passe par un maximum et commence à décroître. Le calcul donne le maximum pour le Ca à la température $T \approx 5000°$, pour le Ti à $T \approx 5500°$ *.

4. Le rôle de l'ionisation et, par conséquent, de la température ne finit pas là. Grâce au champ qui agit le long de l'arc, les ions qui se forment dans la colonne sont entrainés vers le cathode; déjà Mankopf ([10]) a montré que la concentration des ions décroît au milieu de l'arc tandis qu'elle augmente dans la région du cathode. Les raies de la majorité des éléments sont considérablement plus fortes auprès du cathode qu'au milieu de l'arc; ce renforcement, d'après Mankopf, est d'autant plus grand que le potentiel d'ionisation de l'élément donné est plus petit. Mais le nombre d'atomes ionisés étant lié au nombre d'atomes neutres par la relation invariable (2), l'appauvrissement des parties moyennes de l'arc en ions doit s'accompagner d'un appauvrissement en atomes neutres. Nos formules (3) ne sont pas, par conséquent, complètes; car elles ne tiennent pas compte de l'influence du champ. Nous donnons ci-dessous le calcul approximatif de l'intensité des raies en égard à l'ionisation thermique et au champ. Pour simplifier le problème nous nous bornons à une seule dimension et supposons que la colonne de l'arc a la forme d'un cylindre de longueur L ayant la température T. La matière considérée est émise par les électrodes, à raison de n atomes par unité de temps. Ces atomes diffusent à l'intérieur de la colonne, le coefficient de diffusion étant D (nous le posons le même dans le cas d'ions ou d'atomes neutres). Les atomes sont perdus par évaporation dans l'atmosphère par la surface latérale du cylindre, le nombre d'atomes perdus de cette façon étant proportionnel à leur concentration M. Le processus tant stationnaire, l'on a: $n - kM = 0$.

En unité de temps et de volume αN_0 ions sont formés par ionisation thermique, alors que $\beta N_i N_e$ ions se transorment en atomes neutres par recombinaison. Le nombre d'électrons en unité de volume N_e est supposé déterminé par le gaz essentiel et constant tout le long de l'arc. Un champ E agit le long de l'arc, entraînant les ions à mobilité b vers le cathode, où ils perdent leur charge et reviennent de nouveau dans l'arc ([9]). Pour que le nombre d'atomes neutres et d'ions soit stationnaire on doit avoir:

$$\left.\begin{aligned} \frac{\partial N_i}{\partial t} &= D\frac{\partial^2 N_i}{\partial z^2} + \alpha N_0 - \beta N_i N_e - Eb\frac{\partial N_i}{\partial z} - kN_i = 0, \\ \frac{\partial N_0}{\partial t} &= D\frac{\partial^2 N_0}{\partial z^2} - \alpha N_0 + \beta N_i N_e - kN_0 = 0. \end{aligned}\right\} \quad (4)$$

Les conditions aux limites, exprimant que le flux total des particules des additions auprès des électrodes est égal au nombre émis par les électrodes, ont la forme:

$$D\frac{\partial N}{\partial z} - EbN_i = n \qquad \text{à l'anode} \qquad (z=0),$$

$$D\frac{\partial N}{\partial z} - EbN_i = -n \qquad \text{au cathode} \qquad (z=L).$$

* Ces considérations trouvent un usage étendu dans l'astrophysique pour la classification spectrale des étoiles. Il est curieux de noter que les conditions dans l'arc, en ce qui concerne la température et la pression du gaz électronique, sont proches de celles de la chromosphère du Soleil.

Dans le cas d'absence du champ, ainsi que d'émission et d'évaporation des atomes, les équations (4) deviennent: $\alpha N_0 - \beta N_i N_e = 0$.

Dans ce cas l'équilibre s'établit entre N_i et N_0, ce qui s'exprime par la formule de Saha. Par conséquent $\frac{\alpha}{\beta N_e} = \gamma^\circ$. Mais dans le cas général $\frac{N_e}{N_0} = \gamma$ diffère de γ° et dépend en général de z; on peut démontrer néanmoins que pratiquement on peut poser γ égal à γ° et négliger sa dépendance de z. En effet, αN_0 et $\beta N_i N_e$ excèdent de beaucoup les autres termes de l'équation (4), ce qui exprime le fait que l'équilibre entre les atomes et les ions s'établit avec une rapidité beaucoup plus grande que celle des processus de diffusion. Si l'on pose $\alpha N_0 - \beta N_i N_e = g(\alpha_0 N_0 - \beta_0 N_i N_e)$, g étant un nombre très grand par rapport à l'unité, l'on peut développer N_i et N_0 en séries de puissances inverses de g; les substituant dans l'équation (4) on peut chercher des approximations successives par γ. En première approximation l'on a:

$$\gamma^{(1)} = \gamma_0 \left[1 - \frac{\gamma^\circ}{(1-\gamma^\circ)^3} \left(\frac{Eb}{D} \right)^2 \frac{N_0^{(0)}}{N_0^{(0)} + \frac{1}{g} N_0^{(1)}} \frac{1}{g} \frac{D}{\beta_0 N_e} \right];$$

l'on voit que $\gamma = \gamma^\circ$ avec une précision suffisante. Si l'on adopte cette valeur de γ, on ne doit plus employer l'équation (4); il faut prendre l'équation:

$$D \frac{\partial^2 N}{\partial z^2} - Ebx \frac{\partial N}{\partial z} - kN = 0,$$

moins sensible à la valeur de γ et représentant la somme des deux équations (4). Sa solution est:

$$N(z) = \frac{n}{k} \frac{1}{e^{-\sigma_1 L} - e^{-\sigma_2 L}} \left\{ \sigma_1 e^{-\sigma_1 z} (1 + e^{-\sigma_2 L}) - \sigma_2 e^{-\sigma_2 z} (1 + e^{-\sigma_1 L}) \right\} \qquad (5)$$

où

$$\sigma_1 = -\frac{Ebx}{D} - \sqrt{\frac{1}{4}\left(\frac{Ebx}{D}\right)^2 + \frac{K}{D}}; \quad \sigma_2 = -\frac{Ebx}{D} + \sqrt{\frac{1}{4}\left(\frac{Ebx}{D}\right)^2 + \frac{K}{D}}.$$

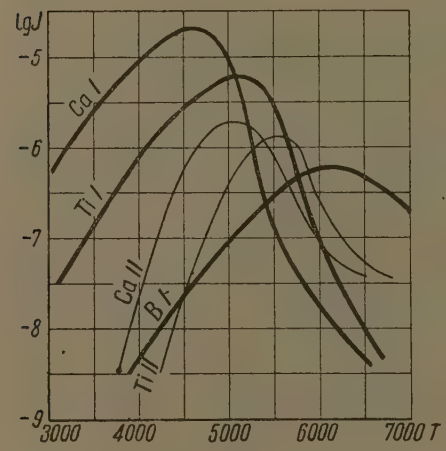

Fig. 1.

Ces considérations, comme il a déjà été signalé, sont approximatives (une dimension, constance de la température à travers la colonne, absence de la composante radiale du champ). En outre, ces considérations ne sont légitimes que pour de petites concentrations des additions.

5. L'expression (5) montre que la distribution obtenue de N en z a la forme barométrique, mais que le coefficient exponentiel renferme le degré d'ionisation, c'est-à-dire qu'il dépend de T. Cette inconstance est la suite de ce que les deux sortes d'atomes N_i et N_0 sont liés par une expression définie (ils «réagissent» entre eux) et cela détermine essentiellement l'aspect du phénomène. L'étude de la distribution de N en z montre que le maximum de N se trouve non pas sur le cathode même, mais à une certaine distance. On voit cela nettement sur nos spectrogrammes et sur ceux de Mankopf. Quant à la dépendance entre N et la température, vu l'absence de données

pour k, nous prenons le cas limite pour tirer au moins des conclusions qualitatives, en supposant l'absence d'émission de la matière pour les électrodes et, par conséquent, l'absence d'évaporation dans l'atmosphère. Il existe des raisons de penser que le rôle de ces facteurs est négligeable ($n \ll M$) et se réduit à l'entretien de l'état stationnaire. Posant $n=0$, $k=0$, $\lim \frac{n}{k} = M$ nous obtenons pour le milieu de l'arc:

$$N = M \frac{Eb}{D} \frac{x}{e^{\frac{Ebx}{D}\frac{L}{2}} - e^{-\frac{Ebx}{D}\frac{L}{2}}}.$$

Cette valeur de N doit être substituée dans (3). Le graphique (1) montre la dépendance de I_r (courbes pleines) et de I'_r (courbes déliées) de T pour le Ca, Ti et B. On supposait que E est proportionnel à T et égal à 25 V/cm pour $T=7000°$. On voit que lorsqu'il est tenu compte du champ, le maximum pour les raies d'arc est plus marqué et un maximum apparaît pour les raies d'étincelle.

6. Des observations qualitatives furent effectuées sur la variation de l'intensité des raies avec la température; l'on introduisait dans l'une des électrodes une quantité considérable de NaCl et l'on photographiait le spectre de l'arc à différents stades, au fur et à mesure que le Na était consumé La variation de la température était constatée par la variation du gradient du champ, qui va parallèlement à la température; cette variation du gradient était constatée par la croissance graduelle de la différence des potentiels aux bornes de l'arc, le courant étant constant. Les spectrogrammes obtenus sont en bon accord sous le rapport qualitatif avec les courbes de la fig. 1. L'étude de la distribution le long de l'axe de l'arc a confirmé de même les considérations développées ci-dessus: à basse température l'intensité des raies est invariable dans toute la longueur, les ions emportés sont compensés aux dépens des atomes neutres. Quand les raies des atomes neutres disparaissent, l'intensité des raies d'étincelle commence à décroître au centre de l'arc et à croître auprès du cathode.

Les considérations exposées, bien qu'approximatives, démontrent qu'il est toujours nécessaire dans les applications pratiques de l'analyse spectroscopique de tenir compte de la déformation éventuelle des résultats par l'introduction, lors d'une analyse, d'une quantité appréciable de matière à bas potentiel d'ionisation. Pour obtenir des conditions de travail toujours reproductibles, il convient d'effectuer toutes les analyses en introduisant supplémentairement dans l'arc une quantité considérable de NaCl ou de KCl, d'autant plus que l'arc brûle dans ces conditions d'une façon plus tranquille.

Nous profitons de l'occasion pour exprimer nos remerciements sincères aux prof. M. A. Leontovitch et G. S. Landsberg pour leurs conseils précieux et leur concours à ce travail.

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GEOPHYSIK

EINIGE CHARAKTERISTISCHE MOMENTE DER HORIZONTALEN VERMISCHUNG DER WASSERMASSEN DES KASPISCHEN MEERES IM SYSTEM $S=f(t)$

Von W. STOCKMANN

(Vorgelegt von O. J. Schmidt, Mitglied der Akademie, d. 9. I. 1937)

Wassermassen oder Wasserkörper im Meere sind, nach A. Defant(1), solche Flüssigkeitsvolumina, welche von mehr oder weniger beträchtlicher Ausdehnung sind und sich voneinander durch charackteristische physikalisch-chemische Eigenschaften unterscheiden. In erster Annäherung werden wir jede Wassermasse als homogen, sowohl hinsichtlich ihrer Temperatur als auch ihres Salzgehaltes, ansehen.

Eine sehr erfolgreiche Methode für das Studium der Turbulenz und der Mischungvorgänge im Meere ist die Herstellung der $S=f(t)$-Kurven(2), die den funktionalen Zusammenhang zwischen dem Salzgehalt S des Meereswassers und dessen Temperatur t für verschiedene Tiefen eines vertikalen Schnittes der Hydrosphäre angeben. Nach einem Grundprinzip liegen die Mischungsprodukte zweier Wassermassen auf der Geraden, die durch die zwei Punkte, welche in dem System $S=f(t)$ zwei homogenen Wassermassen entsprechen, geht. Gewöhnlich stellen die t—S-Kurven, die bei vorgegebener vertikaler Temperaturverteilung und Salzgehalt im Meere hergestellt sind, gebrochene Linien dar. Die Wendepunkte dieser Kurven entsprechen den «Kernen» der verschiedenen tief liegenden Wassermassen, während die geradlinigen Abschnitte der t—S-Kurven auf eine Vermischung der Wassermassen in vertikaler Richtung hinweisen.

Das Kaspische Meer—ein kolossaler abgeschlossener Wasserraum mit reicher Zufuhr an Süsswasser und von intensiver Verdunstung—stellt ein ausgezeichnetes natürliches Laboratorium für das Studium der Mischungsvorgänge in vertikaler als auch horizontaler Richtung der Wassermassen dieses Wassergebietes dar.

Infolge der Schubspannungen, die durch Turbulenz hervorgerufen werden und in vertikalen Flächen, die parallel zur Horizontalkomponente der Strömung sind, wirken, sind die horizontalen Mischungsvorgänge der Wassermassen beim Studium geophysikalischer Prozesse in grossem Masstabe besonders interessant. Es liegen Gründe für die Annahme(3) vor, dass diese horizontale Turbulent in dynamischem Sinne eine wichtigere Rolle spielt, als die vertikale, welche gewöhnlich beim Studium der Mischungsvorgänge in der Atmosphäre und der Hydrosphäre berücksichtigt und von den turbulenten Schubspannungen, die in horizontalen Flächen, die der Strömungsachse parallel sind, erzeugt wird.

Es ist leicht zu sehen, dass wir für das qualitative Studium der horizontalen Vermischung der Wassermassen dieselbe t—S-Kurvenmethode benutzen können. In der Tat, wenn ein Vergleich der t—S-Kurven, die für vertikale Schnitte des Meeres hergestellt sind, uns die Möglichkeit gibt, die Ausdehnung der Wassermassen und deren vertikale Vermischung zu beurteilen, so ist es ganz verständlich, dass wir das Studim der Beziehung $S=f(t)$ für verschiedene Punkte irgendeines horizontalen Schnittes der Hydrosphäre aufnehmen können. In diesem Fall muss man im System $S=f(t)$ nur anstelle der Temperatur und des Salzgehalts die verschiedenen Tiefen der betrachteten Vertikale entsprechen, die analogen Elemente für ein und denselben Horizont verschiedener Vertikalen einführen. Der Unterschied zwischen diesem Verfahren und der Methode der vertikalen t—S-Kurven besteht darin, dass die horizontalen t—S-Diagramme nur gestatten die Gebiete der vollständigen Vermischung irgendzweier Wassermassen festzustellen. In der Tat stellt in dem «horizontalen» System $S=f(t)$ jede Gruppe von Punkten, die durch eine gerade Linie verbunden sind, ein Mischungsgebiet dar. In dem horizontalen t—S-Diagramm kann man auch mehrere Mischungsgebiete haben, wenn die Gesamtheit der Punkte auf mehreren Geraden liegt.

Es ist ganz verständlich, dass je mehr Mischungsgebiete (mehrere Geraden) in einem solchen t—S-Diagramme vorhanden sind, desto weniger intensiv die Mischungsvorgänge in horizontaler Richtung sein werden. Wenn die Punkte sehr verstreut liegen und deren Verbindung durch eine Gerade zu schwierig ist, dann wird eine Vermischung der Wassermassen in horizontaler Richtung überhaupt nicht stattfinden.

Auf Grund dieser Erwägungen wurde das ganze Zahlreiche Material, das während der ozeanographischen Forschungsreisen in den zentralen und südlichen Teilen des Kaspischen Meeres in den Jahren 1933—1937 gesammelt worden war, vom Verfasser genau analysiert.

Die Beziehung $S=f(t)$ wurde sowohl für die am Boden liegenden Horizontalflächen der verschiedenen hydrologischen Stationen, unabhängig von der tatsächlichen Tiefe der Proben (diese Tiefen schwankten innerhalb sehr weiter Grenzen), als auch für dieselben Zwischentiefen aller Stationen, die das Kaspische Meer mit Schnitten bedecken, von Verfasser studiert. Besondere Aufmerksamkeit wurde der Analyse der horizontalen t—S-Diagramme, die für die Temperatursprungschicht hergestellt wurden, geschenkt. Wegen der unbedeutenden Veränderung des Salzgehalts längs der Vertikale (der Salzgehalt längs der Vertikale schwankt für die zentralen und südlichen Teile von 0.1 bis 0.2%), bestimmt man die vertikale Stabilität der Schichten des Kaspischen Meeres ausschliesslich aus dem Temperaturgradiente in der gleichen Richtung, welcher im August sehr grosse Werte (1—2° pro Meter) erreicht.

Die Tiefe des Temperatursprunges im Kaspischen Meere schwankt zwischen 25 und 50 m. Das Vorhandensein einer so scharf ausgeprägten vertikalen Stabilität der Schichten des Kaspischen Meeres macht das Studium des Einflusses dieses Faktors auf die Mischungsvorgänge der Wassermassen in horizontaler Richtung besonders interessant.

An den von uns hergestellten horizontalen t—S-Diagrammen war eine mehr oder weniger beträchtliche Dispersion der Punkte in der S, t-Ebene zu beobachten. In den meisten Fällen waren mehrere Mischungsgebiete der Wassermassen in den zentralen und südlichen Teilen des Meeres festzustellen. Die räumliche Verteilung dieser Mischungsgebiete war hingegen völlig gesetzmässig. Ein Vergleich der Diagramme mit den darauf eingetragenen Mischungsgebieten, die für die Schichten mit maximaler (Schicht des Temperatursprunges) und minimaler (Schichten am Boden) vertikaler

Stabilität hergestellt wurden, ermöglichte uns eine wichtige Schlussfolgerung hinsichtlich des Zusammenhanges zwischen der vertikalen Stabilität und den Prozessen der horizontalen Vermischung der Wassermassen zu ziehen. Die Streuung der Punkte in den t—S-Diagrammen, die für die Tiefen mit maximaler vertikaler Stabilität hergestellt wurden, war immer viel grösser als die Streuung der Punkte für die Tiefen mit minimaler Stabilität. Die S—O_2-Diagramme gaben ähnliche Resultate. Infolgedessen war die Anzahl verschiedener Mischunhgsgebiete in den Schichten mit minimaler vertikalen Stabilität immer grösser als die der Mischungsgebiete in der Tiefe des Temperatursprunges.

Auf Grund der eben erwähnten Tatsachen, kamen wir zu dem Schluss, dass die Existenz der vertikalen Stabilität eine Steigerung der

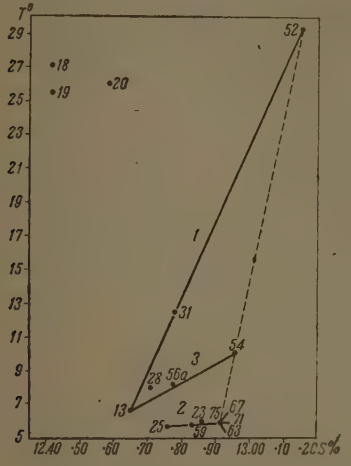

Fig. 1.

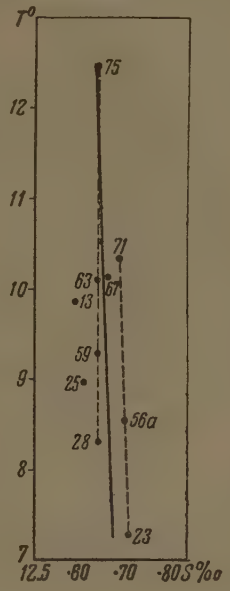

Fig. 2.

Intensität der Wassermassenvermischung in horizontaler Richtung längs isopyknischer Flächen bewirken muss. Die vertikale Stabilität wirkt, bekanntlich, auf die Intensität des turbulenten Austausches in vertikaler Richtung in ganz bestimmter Weise—der vertikale Austausch nimmt in diesem Falle ab. Dies hat schon V. Walfrid Ekman (4) in seinen klassischen Versuchen [und später H. Pettersson (5) und H. Malmheden] gezeigt. Wir haben jedoch bis jetzt keine vollständige Lösung der Frage über den Zusammenhang der vertikalen Stabilität mit der horizontalen Vermischung.

G. I. Taylor (6), zum Beispiel, nimmt in vielen seinen Untersuchungen an, dass die vertikale Stabilität in der Atmosphäre die Komponenten der turbulenten Vermischung isotrop verringert. Nach Taylor degeneriert «die Kugel der turbulenten Vemischung», die in einer homogenen Flüssigkeit stattfindet, in eine Kugel von kleinerem Radius, wenn vertikale Stabilität vorhanden ist.

Unsere Folgerungen stehen im Widerspruch zu einem solchen Schluss. Es ist ganz klar, dass die Kugel der turbulenten Vermischung bei Vorhandensein vertikaler Stabilität und von Wirbelbewegungen in grossen Ausmasse, die zur Vermischung sehr ausgedehnter Wassermassen im Meere führen, in ein Ellipsoid, dessen kleine Achse den isopyknen Flächen parallel ist, degeneriert. Diese

Folgerung bestätigt vollkommen eine ähnliche Meinung von A. Parr (7), welcher aber eine etwas andere Methode bei dem Studium dieser Frage benutzte. Hierbei ist auf die Fehlerhaftigkeit der Annahme von Parr (7) hinzuweisen, der annimmt, dass die grosse Achse des Austauschellipsoids den isopyknen Flächen parallel ist. Diese Annahme von Parr kann mit Hilfe der tensoriellen Turbulenztheorie widergelegt werden. Wir bedauern sehr, dass wegen des beschränkten Umfanges dieses Aufsatzes wir keine Mö-

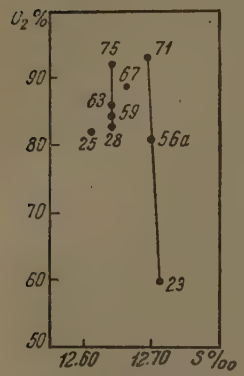

Fig. 3.

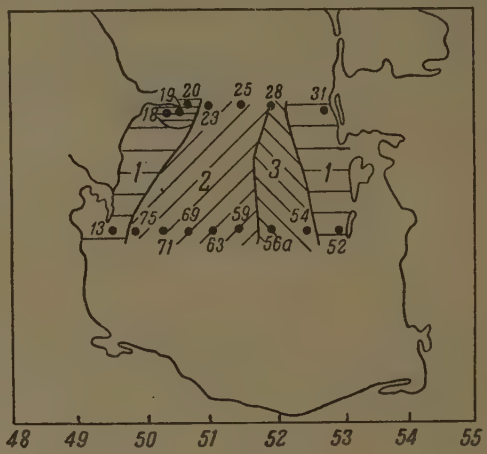

Fig. 4.

glichkeit haben, das ganze zahlreiche graphische Material, welches so überzeugend die dargelegten Erwägungen illustriert, hier anzuführen (dieses Material wird nächstens veröffentlicht werden).

Wir geben hier nur einen charakteristischen Fall der Horizontalmischung des Wassers im südlichen Teil des Meeres im Sommer 1933 an. Fig. 1 zeigt *t*—*S*-Diagramm, das nach den Angaben der Untersuchungen an Bodenschichten zusammengestellt ist (darauf sind drei Mischungsgebiete bezeichnet). Fig. 2 gibt ein ähnliches Diagramm für die Tiefen mit der maximalen Vertikalstabilität. Wie aus Fig. 1 und 2 zu ersehen ist, sind die Vorgänge der Horizontalmischung in einer Tiefe mit der maximalen Vertikalstabilität viel intensiver im Vergleich zu der Horizontalmischung in Bodenschichten. Diese Tatsache wird durch das S—O_2-Diagramm bestätigt, das für die Tiefen mit der maximalen Vertikalstabilität aufgebaut ist. Auf dem letzten Diagramm (Fig. 3) sind dieselben Gebiete der Horizontalmischung des Wassers, die in Fig. 2 angegeben sind, deutlich zu ersehen.

Fig. 4 zeigt die räumliche Verteilung der Mischungsgebiete in Bodenschichten des Meeres.

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BIOPHYSIK

ZUR THEORIE DER ELEKTRISCHEN LEITFÄHIGKEIT SICH POLARISIERENDER OBJEKTE

Von N. N. MALOV

(Vorgelegt von S. I. Wawilow, Mitglied der Akademie, d. 4. II. 1938)

Die Erscheinungen, die man bei dem Stromdurchgang durch die lebenden Gewebe beobachtet, insbesondere die Frequenzabhängigkeit der Reizschwelle und der Impedanz (die Impedanz besteht hier aus einer aktiven und einer kapazitätiven Komponente), zeigen, dass an den Grenzflächen der Gewebe eine starke Polarisation stattfindet.

Die von Nernst[1] entwickelte Theorie, die nur die Diffusionserscheinungen berücksichtigt, gibt das allbekannte Reizungsgesetz:

$$I = p\sqrt{f},$$

welches aber nur bei tiefen Frequenzen richtig ist[2] (f—die Frequenz, I—die Stromstärke).

Die Berechnung des Widerstandes r und der Kapazität K nach dieser Theorie [3,4] gibt:

$$r = \frac{r_0}{\sqrt{f}}, \quad K = \frac{K_0}{\sqrt{f}}.$$

In weitem Frequenzbereich sind auch diese Abhängigkeiten unrichtig.

Neulich habe ich gezeigt[5], dass man eine gute Übereinstimmung der Reiztheorie mit der Erfahrung erhalten kann, wenn ausser den Diffusionserscheinungen auch die Bildung der Doppelschichten berücksichtigt wird.

In ähnlicher Weise ist es möglich die komplizierten Frequenzabhängigkeiten des Gewebewiderstandes zu erklären.

Betrachten wir eine Grenzfläche, durch welche ein Strom

$$I = I_0 \sin 2\pi ft \qquad (1)$$

fliesst. An der Grenzfläche findet eine Konzentrationsänderung statt, die zum Teil durch die Diffusion kompensiert wird. Weiter ist zu berücksichtigen, dass ein Teil der Ionen für die Bildung der Doppelschicht notwendig ist[6], und ein anderer Teil (den wir proportional der Konzentrationsänderung annehmen können) durch die Grenzfläche übergeht, da in den Geweben vollständig undurchlässige Grenzen kaum vorhandensein können.

Weiter wir nehmen an, dass der Abstand zwischen zwei Grenzflächen gross ist.

Um bei diesen Voraussetzungen die Konzentration an der Grenzfläche ($x=0$) zu berechnen, haben wir die Diffusionsgleichung:

$$\frac{\partial c}{\partial t} = k \frac{\partial^2 c}{\partial x^2}, \tag{2}$$

zu integrieren. Die Grenzbedingungen seien folgende:

1) bei $x=\infty$ muss $c=c_0$ sein bei allen t (c_0—die Anfangskonzentration);
2) bei $x=0$ muss

$$mI_0 \sin 2\pi ft = -k \frac{\partial c}{\partial x} + s \frac{\partial c}{\partial t} + p(c - c_0) \tag{3}$$

(m, s, p—Konstanten) sein.

Die Lösung hat die Form:

$$c = c_0 - Be^{-\frac{\xi}{x}} \cos\left(2\pi ft - \frac{x}{\xi} + \vartheta\right), \tag{4}$$

wo

$$\xi = \sqrt{\frac{k}{\pi f}}, \tag{5}$$

$$\operatorname{tg}\vartheta = \frac{n \sin\vartheta}{n\cos\vartheta} = \frac{\alpha_0}{\beta_0} = \frac{1 + \frac{p}{k}\xi}{1 + \frac{2\pi fs}{k}\xi} = \frac{1 + \frac{M}{\sqrt{f}}}{1 + N\sqrt{f}}, \tag{6}$$

$$B = \frac{m}{k}\xi \frac{I_0}{\sqrt{\alpha_0^2 + \beta_0^2}} = \frac{B_0 I_0}{n\sqrt{f}}. \tag{7}$$

Da die Polarisationselektromotorische Kraft gleich

$$e = A_0 \ln \frac{c}{c_0} = A_0 \ln\left(1 + \frac{\Delta c}{c_0}\right) \tag{8}$$

ist, so erhalten wir bei nicht zu grossen Stromstärken (wenn $x=0$):

$$e = e_1\left[1 - \frac{A_0 B_0 I_0}{2c_0 n\sqrt{f}} \cos(2\pi ft + \vartheta) + \ldots\right]. \tag{9}$$

Bei ganz schwachen Strömen, die die Reizschwelle nicht übersteigen, ist es möglich nur das erste Glied in der Gleichung (9) beizubehalten. Also finden wir:

$$e = e_1 = -A \frac{I_0}{n\sqrt{f}} \cos(2\pi ft + \vartheta), \tag{10}$$

wo A eine Konstante bedeutet. Ausser der Grenzfläche haben wir die zwischenliegenden Teile der Gewebe zu berücksichtigen, die einen frequenzunabhängigen Widerstand r_∞ darbieten (wenn die Frequenz nicht zu gross ist).

Ist jetzt an dem Objekt eine Spannung U angelegt, so gibt das zweite Gesetz vom Kirchhoff:

$$U = U_0 \sin(2\pi ft - \varphi) = r_\infty I_0 \sin 2\pi ft - e_1 =$$
$$= I_0 \sin 2\pi ft \left(r_\infty + \frac{A}{n\sqrt{f}} \sin\vartheta\right) - I_0 \cos 2\pi ft \frac{A}{n\sqrt{f}} \cos\vartheta. \tag{11}$$

Daraus folgen Gleichungen für den aktiven Widerstand der Grenzfläche

$$r = \frac{n}{2n A\sqrt{f}\cos\vartheta} \tag{12}$$

und die entsprechende Kapazität:

$$K = \frac{n}{2\pi A \sqrt{f} \cos \vartheta}. \tag{13}$$

Die Impedanz der Grenzfläche wird durch die Gleichung

$$Z = \sqrt{r^2 + \left(\frac{1}{2\pi f K}\right)^2} = \frac{A}{n\sqrt{f}} \tag{14}$$

bestimmt.

Die Werte der Koeffizienten M und N in Gleichung (6) hängen von der Intensität der Entwicklung der an der Grenzfläche verlaufenden Prozesse ab, was verschiedene Frequenzabhängigkeiten der Werte Z, r und K für verschiedenen Gewebe ergibt.

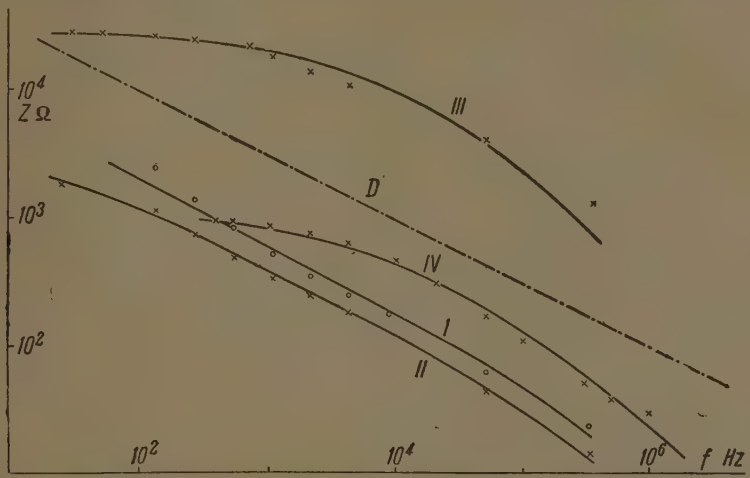

Abb. 1. Die Frequenzabhängigkeit der Impedanz: *I* — Messungen von Lullies (7). Froschhaut (Rückhaut von kleiner *Esculenta*): $r_\infty = 44$ Ohm, $M = 0$, $N = 2 \cdot 10^{-3}$, $A = 2.84 \cdot 10^4$; *II* — Messungen von Lullies (7). Froschhaut (Bauchhaut von kleiner *Esculenta*): $r_\infty = 46$ Ohm, $M = 3$, $N = 2 \cdot 10^{-3}$, $A = 1.925 \cdot 10^4$; *III* — Messungen von Lullies(8). Froschnerv: $r_\infty = 22\,300$ Ohm, $M = 95$, $N = 4 \cdot 10^{-3}$, $A = 2.9 \cdot 10^6$; *IV* — Messungen von Sapengo (9). Froschmuskel: $r_\infty = 105$ Ohm, $M = 80$, $N = 3 \cdot 10^{-3}$, $A = 10^5$.

Leider kann man in der Literatur nur wenige Angaben über die Frequenzabhängigkeit des Gewebewiderstandes in weitem Frequenzbereich finden.

In Abb. 1—3 sind die berechneten Frequenzabhängigkeiten (die Kurven *I*—*IV*) und die entsprechenden Messergebnisse (Kreisen und Kreuzen) dargestellt. Den Verfasser, das Objekt und die Zahlenwerte der Koeffizienten kann man in den Bildunterschriften finden.

Die Kurve *D* ist nach der Diffusionstheorie berechnet.

Die umfangreichen Daten über die Frequenzabhängigkeit des Widerstandes des menschlichen Körpers, die vom Verfasser gemessen wurden (10—14), lassen sich auch mit dieser Theorie verknüpfen. Da aber in diesen Messungen die elektrischen Eigenschaften eines grossen Teils des Körpers (also einer Reihe verschiedener Gewebe) bestimmt wurden, so kann man erwarten, dass eine gute Übereinstimmung mit der Theorie nur die *Z*-Werte ergeben werden, welche von den individuellen Eigenschaften einzelner Gewebe (insbesondere cos ϑ und sin ϑ) nur wenig abhängen. Die entsprechenden Daten sind in Abb. 4 dargestellt.

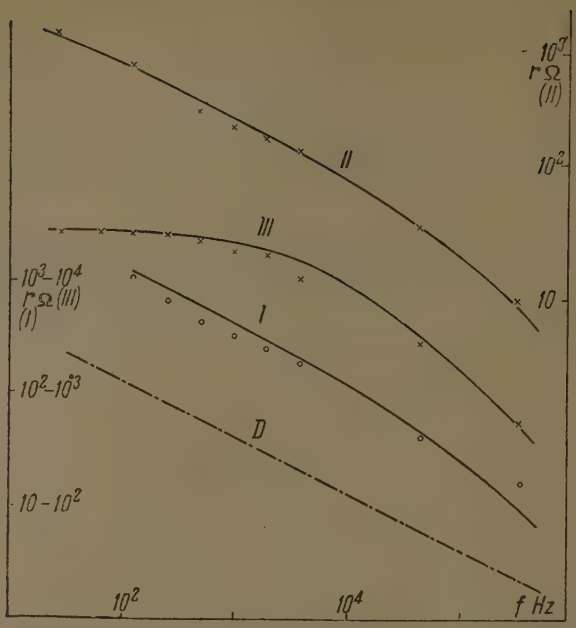

Abb. 2. Frequenzabhängigkeit der aktiven Komponente der Impedanz. Die Bezeichnungen s. in Abb 1..

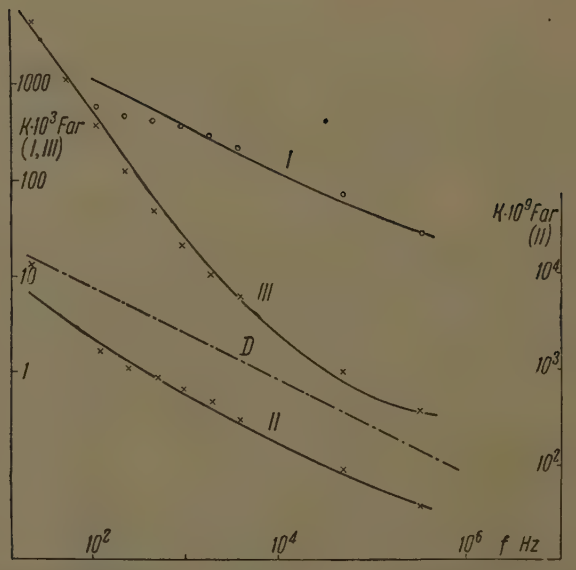

Abb. 3. Frequenzabhängigkeit der kapazitätiven Komponente der Impedanz. Die Bezeichnungen s. in Abb. 1.

Wenn wir berücksichtigen, dass das Frequenzgebiet in allen betrachteten Fällen sehr weit ist und dass die Genauigkeit diesen Messungen unvermeidlich klein ist, so können wir schliessen, dass die entwickelte Theorie eine ziemlich gute Erklärung der Frequenzabhängigkeiten der Impedanz und ihrer Bestandteile gibt.

Die Werte von M, N und A, die die Entwicklung einzelner Polarisationsprozesse in verschiedenen Gewebe kennzeichnen, können ein gewisses Interesse für die Physiologie darstellen.

Zum Schluss ist zu bemerken, dass man die Verkleinerung der Polarisation (also auch Z-Werte) bei starken Strömen, auf welche die Gleichung (9) schliessen lässt, nur bei tiefen Frequenzen beobachten kann (der Theorie entsprechend) ([11]).

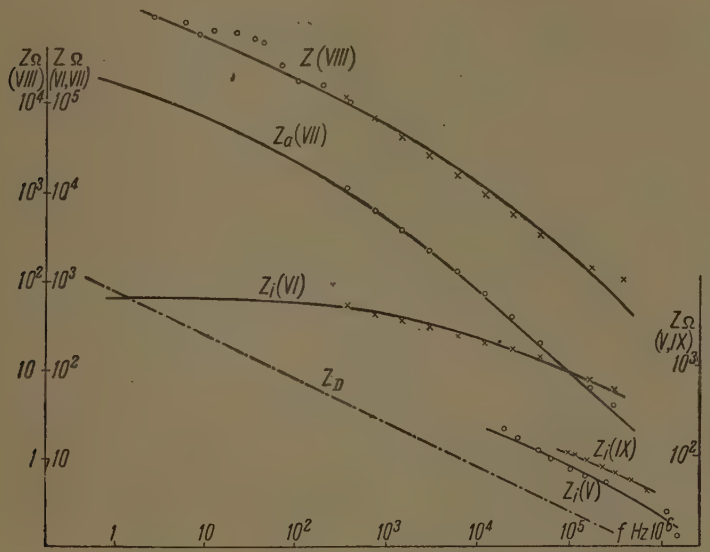

Abb. 4. Frequenzabhängigkeiten der Impedanz des menschlichen Körpers. *V*—Elektroden an dem Unterarm([10]): $r_\infty = 274$ Ohm, $M = 80$, $N = 10^{-3}$, $A = 4.69 \cdot 10^4$; *VI*—Elektroden an den Handflächen (innere Impedanz) ([11]): $r_\infty = 500$ Ohm, $M = 65$, $N = 0$, $A = 4.20 \cdot 10^4$; *VII*— Elektroden an den Handflächen([11]) (Hautimpedanz): $r_\infty = 65$ Ohm, $M = 1$, $N = 4.10^{-2}$, $A = 3.9 \cdot 10^5$; *VIII*—Elektroden an den Handflächen([11–13]) (Gesamteimpedanz): $r_\infty = 565$ Ohm, $M = 1$, $N = 1.2 \cdot 10^{-2}$, $A = 2.8 \cdot 10^5$; *IX*—Elektroden an den Handflächen([11]) (innere Impedanz): $r_\infty = 300$ Ohm, $M = 65$, $N = 0$, $A = 5.4 \cdot 10^4$.

Die Abweichungen der Spannungskurve von der Sinusoide wurden sehr deutlich bei sehr starken Strömen, die die Reizschwelle bedeutend übersteigen, und bei ganz tiefen Frequenzen ($f \leqslant 50$Hz) ([13,14]) beobachtet.

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PHYSICAL CHEMISTRY

DEPENDENCE OF STABILIZATION AND WETTING OF PARTICLES SUSPENDED IN OIL MEDIUM ON THE QUANTITY OF SURFACE-ACTIVE SUBSTANCE ADSORBED

By P. KHOMIKOVSKY and P. REHBINDER, Corresponding Member of the Academy

The purpose of this work is to investigate the state of pigments in oil paints (or printing inks), according to the stabilization and consequent flocculaton of the suspension in a hydrocarbon medium, with surface-active substances*. In order to judge, how the pigments change their dispersion when a surface-active stabilizer is added, the adsorption of oleic acid by pigment particles under the same conditions and the change in the molecular nature of the surface of the particles under the influence of adsorptive layers of stabilizer were also studied.

It has already been shown by one of us and his collaborators ([1],[2]) that if surface-active substances are added to a suspension of hydrophile powders in a hydrocarbon medium, the suspension becomes stable. The stabilization increases with the adsorption of the surface-active substance at the interface powder particle—medium and reaches its maximum when the monomolecular adsorption layer of stabilizer becomes saturated ([2]). When this occurs solvate envelopes([4]) are formed on the adsorption layer of orientated molecules of stabilizer, which have been turned into a phase of low polarity by hydrocarbon chains. These envelopes form a transitional layer between the adsorption films and the dispersion phase, and prevent adhesion of suspended particles to one another. It may be supposed that the thickness of the solvate envelopes increases as the adsorptive layer becomes more saturated, and reaches its maximum when the monolayer is completely saturated. If the concentration of the stabilizer is further increased, part of the suspended particles generally flocculate, i. e. the primarily stabilized particles fuse into flakes. This flocculation may be put down to a second adsorption layer, formed by polar groups on the outside or to the formation of colloidal structure in the adsorption layer.

Four oil (printing inks) pigments of various molecular nature** were studied, after having been carefully dried: submadder varnish ($\theta=133°$),

* Hydrocarbon (non-polar) mediums—vaseline oil (Nujol), benzene, toluene, etc. containing different quantities of surface-active substances (e. g. oleic acid) are models of the oil medium in oil paints. These models are less viscous and do not form films, but correspond to oil medium in their polarity (in the magnitude of their interfacial tension at the interface with water).

** Submadder varnish, pink varnish and yellow varnish are obtained by precipitating the following dyes on to aluminium hydroxide: damask pigment 2B, anthosine 5B and naphthol yellow. Prussian blue (a mineral pigment) is obtained by oxidizing ferrous ferrocyanide ($Fe_2Fe(CN)_6$) in acid solution with air.

Dependence of Wetting and Stabilization of Pigment Particles (4 g) in Toluene (100 cm³) on Quantity of Oleic Acid Adsorbed. C — equilibrium concentration of oleic acid in dispersion medium, q (r_1, r_2) — content of particles in weight % according to fractions $r_1 < r < r_2$, A — adsorption of oleic acid by pigments, Θ — contact angles of the wetting of pigments on the interface drop of water — air

Yellow varnish							Prussian blue						
C in millimoles/litre	q (r_1, r_2) %				A mole/g·10^4	Θ degrees	C in millimoles/litre	q (r_1, r_2)				A mole/g·10^4	Θ degrees
	$r < 2\,\mu$	2—5 μ	5—10 μ	10—25 μ				$r < 5\,\mu$ (0.5—5 μ)	5—10 μ	10—25 μ	25—59 μ		
0.00	Complete precipitation in 6 min. at $H = 45$ cm				—	72	0.00	Complete precipitation in 2 min. at $H = 45$ cm				—	66
0.78	Complete precipitation in 8 min. at $H = 45$ cm				0.25	100	0.78	Complete precipitation in 2 min. at $H = 45$ cm				—	—
2.42	25.7	9.7	48.4	16.2	0.28	117	2.94	Complete precipitation in 4 min. at $H = 45$ cm				0.15	67
8.65	29.2	18.5	13.9	38.4	0.50	180							
16.0	30.1	13.9	17.4	38.6	0.58	(Complete non wetting)	—	—	—	—	—	—	—
							16.6	0	13.1	83.1	3.8	0.29	83
32.0	42.8	6.2	13.9	37.1	0.85	180	33.5	28.3	9.2	62.5	0	0.49	180
48.8	25.7	19.8	26.2	28.3	1.06	180	—	—	—	—	—	—	(Complete non wetting)
66.4	23.7	9.6	23.6	43.1	1.09	126							
100.1	18.2	17.1	15.9	48.8	1.48	123	66.1	31.5	15.8	31.6	21.1	1.02	180
165.0	19.3	13.2	12.9	54.6	1.95	70	101.0	15.3	9.3	51.6	23.8	1.38	—
344.0	27.4	4.7	18.1	49.8	2.78	52	173.0	13.1	11.5	64.5	10.9	1.47	128
							343.0	10.3	13.2	43.4	33.1	1.96	68

pink varnish ($\theta = 124°$), yellow varnish ($\theta = 72°$) and Prussian blue ($\theta = 66°$). θ—the contact angle of drop of water on a layer of pigment in air ([5,6]), characterized wetting ($B = \cos\theta$, Table and Fig. 2). The suspensions (4%) were prepared in toluene or in a mixture of toluene and highly pure vaseline oil.

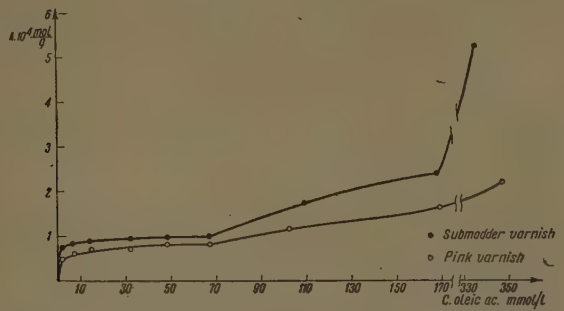

Fig. 1. Adsorption isotherms of oleic acid by pigments.

The pigment dispersion in the suspensions was measured with the vacuum sedimentometer accepted in our laboratory ([1]). The quantity of oleic acid adsorbed by the pigments was determined, after equilibrium had been established, by titrating the dispersion medium with a solution of KOH in alcohol (wiht phenolphthalein) before and after adding the pigment. To determine the change in the molecular nature of pigments, due to the adsorption of oleic acid, the wetting of a layer of pigment particles by a drop of water in air was measured after the method accepted in our laboratory ([5]).

The results of these measurements (table) show that all the pigments are well stabilized with oleic acid in a hydrocarbon medium. If oleic acid is added after stabilization has reached its maximum, the suspension flocculates. The stabilization limit is reached when no more oleic acid can be adsorbed by the pigments. This happens, apparently, when the monomolecular adsorption layer is saturated (inflection in the adsorption isotherms; Fig. 1). The oleic acid adsorption limit corresponds as well to a minimum wetting of the particles with water, i. e. the maximum of hydrophobity ($B = -1$) (Fig. 2). An increase in the concentration of the stabilizer further increases the adsorption. This is obviously due to the chemically free polymolecular adsorptive layer. The pigment particles again show an increase in their wetability (owing to their being covered with thick films of oleic acid), and a pronounced flocculation.

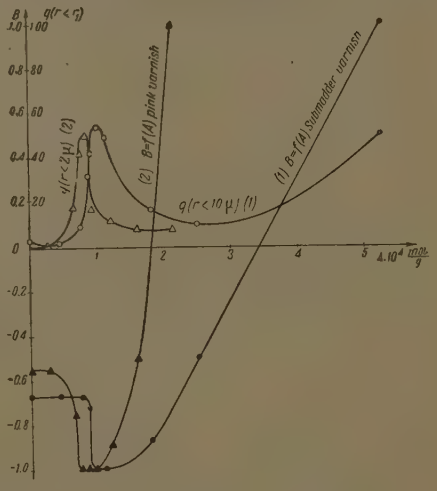

Fig. 2. The dependence of the wetability and the stabilization of pigments on the quantity of oleic acid adsorbed.

Fig. 2 shows how the wetability of pigment particles and the quantity of small fraction of particles which characterizes stabilization, are changing with the oleic acid adsorption (a).

As seen from the Fig. 2, there exists a simple relation between the stabilization and wetting of the pigment particles, and the quantity of stabilizer adsorbed, i. e. the degree of orientation of the surface-active molecules, which reaches its maximum when the adsorptive monolayer is completely saturated. When the values of the submadder varnish and yellow varnish wetability pass through the inversion point ($B>0$), some peptization is again observed among the flakes of these pigments. It is worthy to note that the surface of the more hydrophobe pigments practically do not change their molecular nature at low concentrations of oleic acid; their wetability changes abruptly only when the adsorption reaches $7—8\times10^{-4}$mole/g. The molecular nature of the surface of more hydrophile pigments (especially yellow varnish) at low concentrations of oleic acid immediately becomes more hydrophobe, i. e. the wetability is decreased.

The full specific surface of the pigments (S_1) was computed from the adsorption values at the saturation point of the monolayer, while the geometric (external) specific surface (S_2), from the results of a dispersion analysis taken at the stabilization limit. The specific surfaces of submadder varnish, pink varnish, yellow varnish and Prussian blue are $S_1=27.9$, 22.3, 27.9 and 39.1; $S_2=0.11$, 0.24, 0.25 and 0.35 m²/g respectively. Thus the full specific surface is approximately 10—20 times greater than the external specific surface.

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CHEMISTRY

ON CLEVE'S TRIAMINE

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The triamine of quadrivalent platinum $[(NH_3)_3PtCl_3]Cl.H_2O$ had heen so superficially studied, that even the number of molecules of its crystallization water had not been determined accurately. This may be explained by the difficulties met with in the synthesis of this compound, which is generally produced by oxidizing the bivalent triamine (NH_3) ClPtCl with chlorine; and the bivalent triamine is obtained with difficulty.

We were successful, at first in collaboration with student L. S. Samodelkina, in developing a method of obtaining the quadrivalent triamine, which excludes the complex process of producing the bivalent one. This method consists in oxidizing with chlorine the solution, obtained from the reaction between one molecule of ammonia and Peirone's salt $(NH_3Cl)_2Pt$. After oxidation the following compounds are formed in the solution: Cleve's salt $(NH_3Cl)_2Cl_2Pt$ (in insignificant quantities), Gro's salt $(NH_3)_4Cl_2PtCl_2$ and the triamine, which can easily be separated from the two former salts by twofold precipitation with hydrochloric acid. Crystals of pure triamine vary greatly in colour according to their size, large crystals being bright dark yellow in colour, and the small ones having only a greenich yellow tint. Both have exactly one molecule of crystallization water. The solubility at 20° is 5.64%. Under the action of alkali, strong enough solutions of the triamine give anhydrous amidodiamine according to the equation:

$$(NH_3)_2NH_3Cl_3PtCl + KOH \rightarrow KCl + H_2O + (NH_3)_2NH_2Cl_3Pt.$$

Amidodiamine dissolves poorly in water, has a brick-red colour when the crystals are large and an orange colour when they are small. Under the prolonged action of alkali it slowly changes into a slightly soluble, white, crystallic substance, whose exact composition has not yet been established. Judging by the quantitative reactions, $(NH_3)_3Cl_3PtOH$ is about as strong a base as magnesium hydroxide.

Similarly to magnesium hydroxide free amidodiamine does not precipitate out under the action of ammonia in the presence of ammonium salts, but in their absence an abundant precipitate is formed when ammonia is added to a saturated solution of triamine chloride or nitrate. Besides triamine chloride, the nitrate and sulphate have also been obtained; both of these salts contain crystallization water. One of the intraspheral chlorines of triamine proved to be quite active. The position of this chlorine was determined as a quite unique one, by substituting it with ammonia (see next article) and with a nitrogroup.

According to the principle of trans-influence, a chlorine in a trans-position to the chlorine should have been substituted when the first nitrogroup was introduced. An isomeric nitrotriamine $(NH_3)_2NH_3NO_2Cl_2PtCl$, which

I had obtained before, should have been produced. This method of producing isomeric mononitrotriamines was used by me for triamines containing intraspheral ethylenediamine and it justified itself in all cases. Experiments on the action of sodium nitrite on Cleve's triamine proved beyond all doubt that the only product of the reaction is an isomer of trans-nitrochlorine. In this case it was wery easy to establish that there was no trace of the symmetrical isomer (obtained before), since the latter gives a quite characteristic and sensitive resinifying reaction under the action of alkali. These two triamines differ greatly:

1. In the behaviour of their crystallization water. When kept in air the symmetric triamine readily loses its crystallization water, and turns anhydrous when precipitated with hydrochloric acid. The new triamine has a composition of $(NH_3)_3ClNO_2ClPtCl.H_2O$, irrespective of whether it is produced by being crystallized out of water or by being precipitated by hydrochloric acid.

2. In their rection with alkalis. The symmetric triamine does not under any conditions give a precipitate of amidodiamine with alkalis. If the solutions are left standing, they characteristically resinify, and a colloidal precipitation of indefinite composition is formed. The new triamine under the action of alkali at once gives a precipitate of an amidodiamine, exactly corresponding in composition to $(NH_3)_2NH_2ClNO_2ClPt$. This substance is slightly soluble in water, has a lemon-yellow colour, gives (in solution) an alkaline reaction (on litmus) and in reaction with acids gives corresponding salts.

On the prolonged action of alkaline bases the solution gradually becomes colourless, and a finely crystallized white precipitate appears in very small quantities. It must be stated that the isomers which I have described, represent a peculiar case of isomerism. This isomerism is a consequence only of the various relative positions of the nitrogroup and the chlorines, while in the inner sphere the charges are similarly situated.

The difference in the behaviour of the isomeric amidodiamines in alkaline solution is characteristic as well of the amidodiamines that have an entirely different composition in the amine part of their molecule, but a similar position of the acid radicals. I observed quite an analogous picture in triamines containing, in their inner sphere, ethylenediamine and ammonia, ethyl, methylamine and even pyridine. It is of interest to note that tribromotriamine, which has a composition exactly analogous to that of the trichloride, viz.: $(NH_3)_3Br_3PtBr.H_2O$, behaves quite differently from trichlorotriamine in its relation to alkalis; no precipitate of tribromoamidodiamine is formed.

Conclusions

1. A convenient method of producing the chloride of Cleve's triamine $(NH_3)_3\ Cl_3PtCl$ has been given.

2. It has been shown that under the action of alkali the salts of Cleve's triamine give precipitates of amidodiamine.

3. When one nitrogroup is introduced, nitrodichlorotriamine is produced.

4. This nitrodichlorotriamine has a non-symmetrical structure; the nitrogroup is in trans-position to chlorines.

5. Non-symmetrical dichlorotriamine gives a slightly soluble amidodiamine, which distinguishes it from its symmetrical isomer.

6. A tribromotriamine of the composition $(NH_3)_3Br_3PtBr.H_2O$ has been obtained.

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CHEMISTRY

ON THE CIS-TETRAAMINE OF QUADRIVALENT PLATINUM

By I. I. ČERNIAJEV, Corresponding Member of the Academy

A. Werner's theory predicts the possible existence of geometric isomerism in the following ammonia compounds of quadrivalent platinum: $[(NH_3)_2PtCl_4]$; $[(NH_3)_3PtCl_3]$; $[(NH_3)_4PtCl_2]Cl_2$. However in practice isomerism has been established only for the first of these compounds, and very long ago at. The absence of isomerism in the tetraamines of quadrivalent platinum gave rise not only to various suppositions on the reason for this absence, but also to doubts on the trans-structure of the well-known Gro's chloride: $[(NH_3)_4Cl_2Pt]Cl_2$. At the same time the method of producing Reise's first base, in which all the four ammonias are in one plane, from the chloride, and also the reproduction of salts of Reise's first base from Gro's salts by reducing them, may have served as proof of the trans-structure of Gro's salt. The reason for the absence of isomers of Gro's salt, according to the theory of trans-influence, could not have been the unstability of this cis-isomer, but was simply the impossibility of obtaining this salt, for the lack of a correct way of synthesizing it.

According to the theory of trans-influence, the way of obtaining the cis-isomer lay in the action of ammonia on Cleve's chloride $(NH_3Cl)_2Cl_2Pt$ or on Gerard's salt $(NH_3)_2Cl_4Pt$. It was hard to realize the first method on account of the difficulty in obtaining Cleve's chloride: the second method of synthesis is inconvenient on account of the slight solubility of Gerard's salt.

I have already shown that the existence of cis-tetraamines of platinum is possible, that they are stable and do not show any tendency to isomerize. However, all these cis-tetraamines had a more complex composition than Gro's salt. It may have been asserted, although it is not very probable, that it is very complexity of the composition, that accounts for the direction of the reaction in the syntheses of tetraamines, and their geometrical construction. It was desirable to accomplish the synthesis of cis-tetraamine not only to fill the gap in Werner's series, but also to prove the principle of trans-influence on the simplest example.

Experiments showed, that under the action of a large excess of ammonia Cleve's triamine gives a good yield of tetraamine chloride which has the same composition as Gro's salt, but differs greatly in its properties. The reaction is somewhat complicated, but Gro's salt is not formed. A little pentamine and some products of an unstudied structure are obtained, these products, however, have no Gro's salt in their composition. The solubility of the new tetraamine in water proved to be quite high, viz. at 20° it is 7.15 per cent; at 30°, 7.75 per cent, while the solubility of Gro's salt is in the

hundredths of one per cent at the same temperatures. The temperature coefficient of solubility is low, and cis-tetraamine chloride has to be purified either by precipitating it again from its aqueous solutions with hydrochloric acid, or by passing through a base. The cis-structure of the tetraamine obtained is established quite univocally by reducing it to bivalent platinum.

The reaction goes according to the equation:

$$(NH_3)_2(NH_3Cl)_2PtCl_2 + 2H \rightarrow (NH_3)_3ClPtCl + HCl + NH_4Cl.$$

On reducing one of the coordinates (NH_3—Cl) is liberated, as in the case of the reduction of other cis-tetraamines. When Gro's salt is reduced, it produces a tetraamine of bivalent platinum $[(NH_3)_4Pt]\,Cl_2$. I could find no trace of this easily detectable salt when cis-tetraamine was reduced. Consequently the reduction of trans-tetraamine is a reversible process, while the cis-tetraamine becomes reduced irreversibly.

The reduction of cis-tetraamine requires much more energetic reducing agents than Gro's salt. The latter may be easily reduced by heating it with oxalates or with hydrazine chloride. The cis-tetraamine can withstand being boiled a long time with hydrazine chloride, without changing. In order to reduce cis-tetraamine I found it necessary to use zinc and hydrochloric acid (hydrogen in the moment of its liberation).

The action of alkali on Gro's salt depends on the anion outside the sphere. The chloride of Gro's salt dissolves in alkali without forming any precipitate; the nitrate gives an amidotriamine of the composition: $(NH_3)_3NH_2Cl_2PtNO_3$. The cis-tetraamine gives the same precipitate irrespective of the nature of the outside-sphere anions. The composition of this precipitate is anomalous, and the following three formulae may represent its structure, if the possibility of geometric isomerism is not taken into consideration:

I. $NH_3NH_2(NH_3Cl)_2PtOH \cdot H_2O$;
II. $(NH_2)_2(NH_3Cl)_2Pt \cdot 2H_2O$;
III. $(NH_3)_2(NH_3Cl)_2Pt(OH)_2$.

I think that the first formula is the most probable one, but it must be stated that, when heated to 120° this substance loses both of its water molecules, which is more in accordance with the second formula. This amido-base dissolves quite poorly in aqueous alkalis, crystallizes excellently, has a slight yellowish tint in small crystals. With acids it forms corresponding salts of tetraamine, which are almost colourless in small crystals and slightly yellow in large ones, and gives a basic reaction on litmus when dissolved in water. Besides tetraamine chloride, its sulphate and nitrate were also obtained. These dissolve well in water and give well-formed crystals.

Conclusions

1. Platinum cis-tetraamine, an isomer of Gro's salt: $(NH_3)_4\,Cl_2\,PtCl_2$, has been produced.
2. Its construction was proved by reducing it to the triamine of bivalent platinum.
3. A great difference has been established in the solubilities of the chlorides of the two isomers.
4. Under the action of alkali the salts of cis-tetraamine give a precipitate of the composition: $NH_3NH_2(NH_3Cl)_2PtOH.H_2O$.

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CHEMISTRY

ON «RUBY» SLAG OF RECREMENT OF ALUMINOTHERMAL PRODUCTION OF METALLIC CHROMIUM

By N. N. KURNAKOV, I. I. KORNILOV and G. B. BOKY

(Communicated by N. S. Kurnakov, Member of the Academy, 29. I. 1938)

During a visit to the Dhzungel (Zestafon) ferro-manganese works (Georgia) it the summer of 1937, the attention of the workers of the Institute of General and Inorganic Chemistry of the USSR Academy of Sciences was attracted by dark-red crystalline slags. These slags are a by-product in the metallic chromium production(1) after the aluminothermal method recently adopted at the Dzhungel works. The row material for this proceeding is chromium oxide, supplied by the Ural Potassium Bichromate Plant. The aluminium used for the aluminothermal reduction of chromium is supplied by the Dneper Aluminium Works. Its analysis is as follows: Aluminium 95—96.8%. Iron 0.50—3.0%. Silicon up to 4%.

The change consists of: Chromium oxide 550 kg. Aluminium (powder) 225 kg. Potassium chlorate 50—55 kg. Lime 50—55 kg.

The reduction of chromium proceeds according to the equation:

$$Cr_2O_3 + 2Al = Al_2O_3 + 2Cr + 109 \text{ c. cal.}$$

The process is carried on in lined iron cylinders; a portion of the charge is heated by a voltaic arc; then, gradually as it heats, the rest of the charge is added. The yield of metallic chromium is 80—85%. An examination of the slag formed in the course of this proceeding showed that the slag is porous, of a dark-red colour and finely crystalline in its lower portion.

In cavities in the central part of the slag, finely developed large crystals have been detected. Both the crystals and the dense parts are of a dark-red colour. They form hexagonal tablets. The size is frequently up to 1½ cm in width and 0.1 cm in thickness. These tabular crystals are attached to each other, forming aggregates of alveolar structure.

A measurement of the optical constants by the immersion method showed that the crystals are uniaxial, negative: $N_m = 1.770$; $N_p = 1.762$.

On the basis of these measurements, according to Larsens's and Berman's determinant(2), in has been found that these constants correspond to the aluminium oxide (corundum). Since our crystals contain an admixture of chromium, they should be named rubies.

According to Dana's textbook (3) the constants for corundum are as follows: $N_m = 1.768$; $N_p = 1.760$, i. e. they closely approach to those found by us.

The plant analyses of the slag and the analysis of separate crystals are given below; the latter were carried out in the Institute of General and Inorganic Chemistry by I. V. Tananaev.

	Analysis made at the Dzhungel Works		Analysis made at the Institute of General and Inorganic Chemistry	
Chromium oxide (Cr_2O_3)	4.94		7.57	
Iron oxide (Fe_2O_3) . . .	3.62	up to	0.1	
Calcium oxide (CaO) .	5.74		3.20	
Silica (SiO_2)	3.40		none	
Magnesium oxide (MgO)	—		0.73	
Aluminium oxide (Al_2O_3)	81.39		88.40	by substraction from 100 the above contents.

The qualitative analysis made in the Institute of General and Inorganic Chemistry showed no other components besides the above-mentioned elements. The crystals of ruby from the slag have been measured also by a goniometer, and presented uniform development of simple forms. Always the pinacoid was developed $\{0001\}$ limited at the edges by rhombohedron faces $\{10\bar{1}1\}$. On one of the crystals a prismatic face $\{11\bar{2}0\}$ was observed.

The general view of the crystals is given in the upper part of Figure 1; in the lower part they are viewed from above.

In the following table the results of measurements are given.

Symbol of simple form	φ	ρ
0001	—	0°
$10\bar{1}1$	90°00′	57°33′
$11\bar{2}0$	60°00′	90°00′

$a : c = 1 : 1.362$.

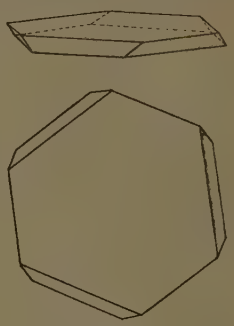

Fig. 1. A ruby crystal from slag.

Judging from the measurement data the crystals should be referred to the hexagonal syngony, the ditrigonal scalenohedric type of symmetry (elements of symmetry $J_6 3\ L_2 3PC$). All these data are very similar to those of natural corundum and rubies.

In Dana's textbook the coordinate $\rho = 57°34'$ is given for the rhombohedron $\{10\bar{1}0\}$. The elements of the crystal, as calculated from this coordinate, are $a : c = 1 : 1.363$.

The optical and geometric constants of natural and synthetic rubies are similar to those of corundum, since the admixture of chromium usually amounts only to some fractions of a per cent. Our rubies are evidence of the fact, that even such a high content of chromium oxide does not markedly affect the constants of the crystals, since an isomorphous replacement of Al_2O_3 by Cr_2O_3 takes place.

A rough determination of hardness according to Mohs scale also showed no considerable deviations from the hardness of corundum.

The crystal samples from slag showed no fluorescence in the rays of a mercury lamp ARK2, but changed their colour from dark-red to bright-red (carmine).

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GENETICS

EVOLUTION OF CULTIVATED FORMS OF GRAPES

By A. M. NEGRUL

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The data we have gathered in examining the species and varieties of grapes in the world collection of the Institute of Plant Industry, and in studying the wild and cultivated forms in the regions of their ancient cultivation (within the USSR), as well as our genetic investigations, have enabled us to make an ecologico-taxonomic classification of grape varieties based on the origin and evolution of this crop plant. The concept of the evolution of cultivated forms of grapes we have elaborated from these data, is given below.

All wild species of the genus *Vitis* may geographically be divided into three groups: the east-Asiatic, the Eurasian and the American. However, despite their strict geographical isolation, they have much in common. All of them are dioecious (some plants bear male flowers, others functionally female flowers with bent stamens and sterile pollen due to degeneration of the vegetative and generative nuclei); the corolla, in distinction from other genera, is shed in the form of a hood; the diploid chromosome number in all species of the subgenus *Euvitis* is 38; the crossability of all species is good, interspecific hybrids having regular meiosis and fertile progeny; there is lacking the passive immunity to phylloxera and fungus diseases possessed by other species of the *Vitaceae*; all species of the genus *Vitis* have small, round fruit, with juicy flesh, sweet juice*, and small seeds with a short beak.

This uniformity of all wild species of the genus *Vitis* may be explained by paleontological data, according to which in the Upper Cretaceous and Tertiary periods forms close to present-day American species (*aestivalis, cordifolia, riparia*) had a common area in the Arctic regions of North-Eastern Asia and North America. These ancient forms of grapes after the separation of the continents presumably gave rise to the present-day species of the genus *Vitis*.

Proceeding from the premise that the geographical separation of species of the genus *Vitis* took place several hundred thousand years ago, it is of interest to establish the essential character and the causes of the differences between these species.

* The capacity of the fruit to accumulate sugar is a characteristic of all wild species of the genus *Vitis* and many species of other genera (*Ampelopsis, Parthenocissus, Ampelocissus*), wich indicates that this quality arose naturally and not as a result of artificial selection. The wild Amur grape, when grown under Central Asiatic conditions, yields as much as 24 per cent sugar, i. e., as much as cultivated varieties.

Analyzing the morphological and biological characteristics of species of the genus *Vitis*, we find that the chief differences concern only biological qualities connected with the ecological conditions of the habitat of these species, Thus, species with a northern area of distribution (*V. labrusca* L., *V. amurensis* Rupr.) are more cold resistant and have a shorter vegetative period than species with a southern area. A striking example of the rôle of natural selection is the acquisition of active immunity by species of grapes growing in the south-eastern part of North America, which is the home of *phylloxera* and many fungus diseases of the grape, while species of other regions (even *V. californica* Bench.), where these diseases and *phylloxera* were unknown until the middle of the XIX century, are susceptible.

Differences between wild species as regards their morphological characters are less marked. They concern only the general habit of the plant and the structure of the leaf.

A somewhat different picture is presented in the case of those few species of grapes which are cultivated for their fruit. These species are more or less polymorphic, depending on how long ago they were introduced into cultivation. The European grape, *V. vinifera* ssp. *sativa* D. C., one of the most ancient forms (over 10 000 years old), has many varieties (more than 3000), which are very diverse as regards morphological and economic characters.

Species of grapes, such as *V. labrusca* L. and *V. rotundifolia* Michx., which were introduced into cultivation comparatively recently (about 150 years ago), are considerably less polymorphic. They are only beginning to accumulate those variations which European grapes already possess. The most ancient form of grape, maintained in Europe from the Tertiary period, is *V. vinifera* ssp. *silvestris* Gmel. The area of this subspecies embraces the ancient Mediterranean region, extending from Spain to Turkmenistan (Kopet-Dagh). This grape is described in Spain, southern France, Italy, in the Rhine, Danube, and Dneper valleys, on the Balkan peninsula, in the Crimea, on the Black Sea coast, in Transcaucasia, and in Kopet-Dagh as a small, round, black fruit with seeds having short beaks. The extensive area of ssp. *silvestris* Gmel. and the repetition of characters common to other wild species of the genus *Vitis* leave no doubt as to the validity of this subspecies. However, if in northern regions it is maintained in its typical form (*typica*), in southern regions, where the course of natural formorigination has proceeded more intensively, it has acquired new formations (hermaphrodite flowers, white or pink fruit, larger fruit, etc.), the nature of which is as yet difficult to disclose.

These aberrant (*aberrans*) types of the wild grape, sharply distinct from other species of the genus *Vitis*, arose, from our point of view, as a result of artificial selection, alternating with processes of natural hybridization, seed sowing, and crossing between seedlings under conditions of wild nature.

The wild grape of the Black Sea coast (the Caucasus), found in districts of ancient Kolkhidia, has to a considerable degree been mixed with the cultivated grape. Here we find all transitions from the real wild grape, ssp. *silvestris* Gmel., to ancient, cultivated local varieties of the wine type. The wild grape of the Kuba district is the most typical. The wild grape of Kopet-Dagh is regarded by us as a hybrid between the wild ssp. *silvestris* Gmel. and the cultivated grape of the East (*orientalis subprol. caspica* Negr.). Here the wild grape, ssp. *silvestris* Gmel., has been maintained in small quantities (25%); it is very possible that it only recently penetrated this region. The wild grape of Western Tian-Shan and Tadzhikistan is an escape of wine varieties introduced here from the Caucasus and widely cultivated in Central Asia prior to the arrival of the Tiurkish tribes. This wild grape has nothing in common with the large-fruited, firm-fleshed varieties now cultivated in Central Asia.

Historical data indicate that the grape was first introduced into cultivation at the time of the development of wine-making in regions of ancient Eastern civilizations. Inasmuch as the habitats of the wild grape, ssp. *silvestris* Gmel., nearest to these regions, are Asia Minor and Transcaucasia, there is full justification to consider that precisely here is the homeland of the cultivated grape. The local origin of a large number of ancient varieties of western Georgia, Kakhetia, and other regions of the Caucasus we now consider as an indisputable fact. The wine varieties with small juicy fruit, with small seeds, and with densely pubescent leaves, in type close to ssp. *silvestris* Gmel., growing in the Black Sea region (Georgia, Asia Minor, Greece, Bulgaria, Hungary, Rumania), we have placed in the ecologico-taxonomic group *prol. pontica* Negr.

Simultaneously with the advancement of civilization westward, grape cultivation and wine-making were introduced into the western lands. Varieties of grapes brought from the Caucasus and Asia Minor to Greece and later to Italy were crossed with local wild forms. In some cases wild forms were introduced directly into cultivation, thanks to which each region developed its own native assortment of grapes. The ancient wine varieties of France, very close to typical ssp. *silvestris* Gmel., characterized by small, compact clusters, small juicy fruits, small seeds, and leaves with a fine, downy pubescence, we have placed in the ecologico-taxonomic group *prol. occidentalis* Negr.

With the extension of viticulture from the Caucasus to the south, under conditions of the intensive, irrigated agriculture of the ancient oases (where the wild grape is unknown), there were created for the grape conditions of isolation, which, combined with the repeated sowing of seeds and the selection of bud mutations, facilitated the accumulation of recessive characters. The first stage of such selection gave, in regions bordering on the Caspian Sea, wine and table varieties (with smooth leaves, fairly large, often branched clusters, and medium-sized, juicy fruit), which are found to this day in Daghestan, Azerbaidzhan, and Turkmenistan. This group of varieties, which is regarded by us as transitional to the large-fruited, firm-fleshed varieties of the East, we have classified as *prol. orientalis subprol. caspica* Negr. The second stage is linked with a later period of grape breeding during the Islam era, when from the first group there were brought out and widely propagated firm-fleshed, large-fruited table varieties, and also seedless varieties of grapes; these have been classified by us as *prol. orientalis subprol. antasiatica* Negr.

The genetic data on the inbreeding and hybridization of varieties belonging to the ecologico-taxonomic groups established by us confirm the autonomy of these groups and also shed light up on their origin. Varieties of the group *occidentalis* and *pontica*, as the closest to the wild grape, ssp. *silvestris* Gmel., possess numerous dominant characters (pubescent leaves, juicy flesh, small round fruit, short vegetative period, etc.). On the other hand, varieties of the group *orientalis*, which arose under conditions of isolation in the ancient oases, possess an abundance of recessive characters (smooth leaves, long, or irregular-shaped fruit, firm flesh, long vegetative period).

The evolution of the large-clustered, large- and long-fruited varieties of grapes we consider to have been a gradual process of selection, under conditions of intensive cultivation of the ancient oases, of large fruited bud mutations and recessives (when sowing seed), with subsequent hybridization of the selected forms. That the large-fruited varieties arose by the gradual accumulation of multiple factors of size and not as a result of a single mutation is shown by inbreeding these varieties. Thus, inbreeding the long- and large-fruited varieties Shiradsouli (Tavris), Husain, etc. gives seedlings with all

transitions of size and shape of fruit: from medium and round to large and long, as in the maternal forms. Crossing large-fruited with small-fruited varieties gives an F_1 with mostly small-fruited types.

Hence, by gradual selection of bud mutations and various combinations there were developed, during the course of many centuries, new qualities, new quantitative, economic characters. which accounts for the remarkable polymorphism of cultivated varieties of grapes.

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AGROPHYSIOLOGY

WINTERHARDINESS OF PLANTS AS INFLUENCED BY YAROVIZATION OF RIPENING SEED-GRAINS

By I. A. KOSTIUČENKO

(Communicated by N. I. Vaviloo, Member of the Academy, 29. I. 1938)

Acad. T. D. Lysenko (4), author of the stage development theory, has shown on a number of farm crops that some characters and properties are as well expressed in the plant as they have been promoted by the conditions of its stage development. Thus, the winterhardiness, one of the economically valuable properties, is closely connected with the length of the yarovization period. This statement has been supported by the subsequent works of other investigators, as Kuperman (3), Pančenko (6), Timofeeva (7) and others (5).

The author of the present paper has established in his previous works (1, 2), that under favourable temperature conditions an embryo already formed, but not yet at rest, i. e. such as of the seed that has begun ripening, but is not yet ripe, is able to pass the yarovization stage on the mother plant just as the embryo of a germinating seed.

This has been confirmed by the experiments with rye carried on by the english scietists Gregory and Purvis (8) under artificial conditions.

We came therefore to the conclusion that seeds reproduced in the north and ripened under the action of low temperatures, having thus passed the yarovization stage during their ripening, will give plants poorly resistant to frost and occasionally unable to stand even a moderate winter.

Table 1

Crop and variety	Locality and year of reproduction	Number of plants		Percentage of plants that have stood the winter of 1936—37
		in autumn	in spring	
Winter wheat:				
Ukrainka var. *erythrospermum*	Derbent, 1936	128	85	66.41
» » »	Pushkin, 1936	186	152	81.72
» » »	Chibiny, 1935	170	37	21.77
» » »	Kamchatka, 1935 . .	63	13	20.97
Winter rye:				
Bezentchukskaya	Bezenchuk, 1936 . . .	128	116	94.31
»	Pushkin, 1936	156	156	100.90
»	Kamchatka, 1935 . .	191	87	71.90

The data in Table 1 are derived from the experiments on the hibernation of some varieties of wheat and rye in Pushkin, 1936—1937. The plants were grown from the seeds, reproduced in different points of the USSR as well as from seeds ripened in the same locality (Pushkin) at different dates of the year 1935.

As seen from the table, the plants of the winter wheat variety Ukrainka, grown from the seeds reproduced in the northern regions (Chibiny, Kamchatka), are three-four times less resistant to cold, compared to the plants grown from the seeds reproduced in Derbent and Pushkin. As to the latter we may note that the seeds reproduced in Pushkin gave a somewhat higher percentage of the plants that had outlived winter than those reproduced in Derbent. In our opinion this should be accounted for by the mild winter conditions of Derbent, which do not direct the natural selection towards the rise of the winterhardiness. Thus, Ukrainka being year by year sown in Derbent not only was not increasing its resistance to cold, but even gradually lost it.

Fig. 1. Winter wheat Ukrainka var. *erythrospermum*. Crop from winter sowing as influenced by the conditions the material was planted under. From left to right: 1) plants from seeds of Derbent reproduction (endured wintering 66.41%); 2) plants from seeds of Pushkin reproduction (endured wintering 81.72%); 3) plants from seeds of Chibiny reproduction (endured wintering 21.77%) and 4) plants from seeds of Kamchatka reproduction (endured wintering 20.79%).

The same relations though not so pronounced are seen in plants of the rye Bezenchukskaya.

The conditions of seed reproduction have of course determined not only the percentage of the plants survived but also the yield in general. This dependence is shown in Fig. 1. In Fig. 2 the temperature conditions are shown under which the seeds for test were forming and ripening at Pushkin and in Kamchatka.

Table 2 presents data on four winter wheat varieties that have stood winter. They grew from seeds harvested in Pushkin at different dates of the year 1935.

The seeds gathered for the first variant of the test ripened earlier because they were taken from plants whose seeds before being sown had past a longer period of yarovization. The seeds of the second variant belongs to plants of the same four varieties but they had ripened 2—3 weeks later.

The data of Table 2 support the statement that seeds ripening under low temperatures afterwards give plants with reduced winterhardiness.

In this connection we can point out that in Sweden, Norway and Finland the seed-grains of winter cereals are cultivated in the southern regions of these countries.

Table 2

Percentage of Winter Wheat Plants that Outlived Winter Being Grown from Seeds Ripened at Different Dates of One Year in the Same Locality (1935, Pushkin)

Variety	Conditions previous to the reproduction of the seeds used for experiments	Number of plants		Percentage of plants that outlived the winter of 1936—1937
		in autumn	in spring	
Ukrainka	50 days yarovization at 2 to 4°C before spring sowing in 1935 . . .	112	32	28.57
Ukrainka	20 days yarovization at 2 to 4°C before spring sowing in 1935 . . .	123	10	8.13
Durable	30 days yarovization at 2 to 4°C before spring sowing in 1935 . . .	64	9	14.06
»	20 days yarovization at 2 to 4°C before spring sowing in 1935 . . .	80	0	0.00
Saratov 0329	50 days yarovization at 2 to 4°C before spring sowing in 1935 . . .	75	23	30.67
» 0329	30 days yarovization at 2 to 4°C before spring sowing in 1935 . . .	52	5	17.31
0256	50 days yarovization at—1°C before spring sowing in 1935	81	41	50.62
0256	40 days yarovization at—1°C before spring sowing in 1935	110	22	20.00

Conclusions

1. When winter wheats ripen during the period of reduced temperatures in summer and autumn (under the average of 14—15° C for 24 hours) the seed-grains pass the whole or a part of the yarovization stage on the mother plant according as their stage is less or more long in the given variety.

2. Plants grown from the seeds which had ripened under reduced temperatures are by far (sometimes 3 to 4 times) not so resistant to cold as the plants reared from the seeds ripened under higher temperatures.

3. The yield in general is considerably decreased in the plants grown from seeds which had passed the whole or a part of the yarovization stage.

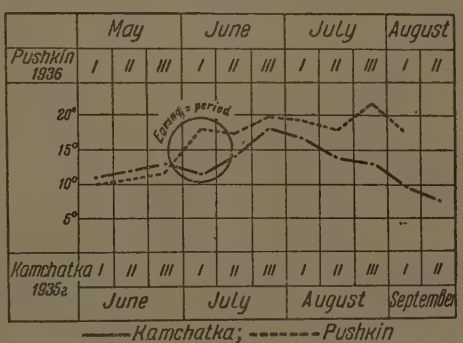

Fig. 2. Average temperatures according to decades for May — August 1936 at Pushkin, and for June—September 1935 at Kamchatka. Seeds frow crops collected at these periods and regions were taken for experiments.

4. Seed grains of winter crops, especially wheat, to be sown in the northern region of the Union must have passed the ripening stage under high temperatures (above 14—15°C as the average for 24 hours).

5. In northern localities each variety of winter crops, especially the first introduced, must be tested on two variants, both the seed-grains of local origin and those from southern regions being involved in the test.

6. In order to introduce the cultivation of winter wheat to Farnorth and north regions of the Union, where till now there exists no sorts of this crop, it seems necessary to organize in adjacent southern regions a cultivation of seed-sorts which yield higher crops immediately after their importation.

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AGROPHYSIOLOGY

ACTION OF SOIL DROUGHT UPON THE FORMATION OF GENERATIVE ORGANS IN SPRING WHEATS

By G. V. ZABLUDA

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The influence of soil drought in spring wheat during the interval between «tillering» and blossoming results in a great diversity of the forms of underdeveloped spikes or of their separate parts ([1], [2], [3], [4]). This results from the fact that the plants at different periods of the formation and growth of the generative organs react in a different way to the conditions of soil drought([5]).

The purpose of the present work was to establish the periods of formation and growth during which generative organs are subjected to a definite degree of injury under soil drought conditions, as well as the physiological causes underlying these phenomena. As early as in 1935 preliminary observations of the process of formation and growth of the generative organs were carried out. We succeeded in establishing that the entire course of formation and growth of the generative organs, from the appearance of the spike embryo till flowering, may be divided into three specific periods; this was taken into consideration when studying the influence of drought on the formation of generative organs.

The work was carried out in 1937, at the Chuvash Institute of Agriculture on spring wheat *Lutescens* 062. Sowing was done into earth vessels, containing 4 kg of absolutely dry soil, 6 plants per vessel. Soil was moistened to 65 p. c. of its moisture capacity. Drought was brought about by elimination of watering: first at the period of the embryonal development of the spike and flower organs, which coincides in time with the period of «tillering» (Ist series); secondly at the period of inner differentiation and formation of flower organs, which coincides in time with the first two; thirds of the phase of stalk-formation (IInd series); further at the period of the most intence growth and ripening of the flower organs, which coincides in time with the last third of the phase of stalk-formation, until blossoming (IIIrd series); and finally during the period of the formation of grain (IVth series). During each period the plants remained unwatered for 10 — 12 days, and in the wilting state—for 5 days.

Observations on the course of formation and growth of generative organs, as well as on the stem growth have shown that the embryonal development of the ear and of the flower organs (Ist series) was not suspended during drought; the growth of the stem and spike in the experimental plants begins somewhat later as in the control. Inner differentiation and the formation of the flower organs (IInd series) also was not suspended during drought, but towards the end of drought the length of the ear in the experimental plant was 1.2 to

1.5 cm, and that of the stem—15 to 18 cm, whereas the control had an ear length of 2.0 to 2.5 cm, and the stalk—22 to 26 cm. Neither did soil drought retard the ripening of the flower organs, yet it delayed their growth (IIIrd series). Towards the end of drought in the experimental plants (IIIrd series), the majority of pollen cells were smaller, of a lighter colour, and not so full-formed as in the control; the length of the spike was 6 to 7 cm, and that of the stem 35 to 40 cm; whereas in the control the spike equaled 8 to 9 cm, and the stalk from 60 to 65 cm. During drought one to three lower leaves became yellow and died away.

Table 1

Action of Drought on the Formation and Growth of the Generative Organs in Wheats

Plant series	Phase of generative organs development during the period of drought action	Number of spike bearing stalks per vessel		Length of spike in cm		Number of spikelets in one spike		Number of undeveloped and sterile spikelets in one spike	
		Total	Appeared after drought	Main stalk	Collateral stalk	Main stalk	Collateral stalk	Main stalk	Collateral stalk
Control		10.5	—	9.0	8.5	14.2	15.1	0.8	3.7
I	Period of embryonal development of the ear and flower organs	18.0	9.0	8.0	6.5	12.2	12.7	0.8	3.2
II	Period of internal differentiation and formation of flower organs	12.5	2.5	8.0	7.0	14.4	13.6	4.3	6.7
III	Period of the most intense growth and ripening of flower organs	11.0	—	8.5	6.0	14.5	14.7	8.0	8.8
IV	Period of grain formation	10.2	—	9.0	8.0	14.2	15.0	0.6	3.6

Data of Table 1 show the way the influence of drought tells on the further formation and growth of the generative organs. The most essential feature is the total number of spikelets in the spike and the number of undeveloped and sterile spikelets. Drought during the period of the embryonal development of the spike and of the flower organs resulted in a reduction of the total number of spikelets in the spike; nevertheless, the number of undeveloped and sterile spikelets greatly increased, attaining in the plants of series III a maximum value of 53.0 to 55.1 p. c. of the total number of spikelets in a spike.

From Table 2 it may be seen that in the plants of series III the yield has decreased at the expense of the reduction of the total number of grains only while the absolute weight has increased. In plants of the IInd series, the yield is lower at the expense of the total number of grains per vessel, as well as at the expense of a decrease of the absolute weight of the grains of the collateral stalks. The plants of the Ist series yeilded a total number of grains per vessel, 35.7 p. c. more than the control, but with a lowered absolute weight of the grain obtained from the collateral stalks. It should be noted that the plants were harvested after the main stalk had ripened, while the collateral ones,

Table 2

Action of Drought on the Yield of Wheat Grain

Plant series	Phase of development of the generative organs during the influence of drought	Number of grains per vessel		Number of grains in ear		Weight of grain per vessel		Average weight of one grain in mg	
		Main stalk	Collateral stalk	Main stalk	Collateral stalk	Main stalk	Collateral stalk	Main stalk	Collateral stalk
Control		199	95	32.2	21.1	7.22	2.97	36.3	31.2
I	Period of embryonal development of the ear and flower organs	152	247	25.3	20.6	5.26	4.60	34.6	18.6
II	Period of internal differentiation and formation of flower organs	109	118	18.2	18.1	3.92	2.69	36.0	17.5
III	Period of the most intense growth and of ripening of the flower organs . . .	85	22	14.2	4.4	3.98	0.89	46.8	40.5
IV	Period of grain formation	192	76	32.0	18.0	5.20	1.52	27.1	20.0

which appeared after drought, had been in the stage of milk ripeness (Ist series) or else in the stage of swelling (IInd series); this accounts for the drop of absolute weight of grain in the collateral stalks of the plants in series I and II.

Finally, phenomena of particular interest accompanying redistribution of plastic substances within the plant at the period of the most intense growth of the generative organs and stalk, i. e. during the last third of the phase of stalk formation until flowering, should be noted. Data of Table 3 show that the spike and the upper internodes of the stalk consume so much nutritive products that the production of the day's assimilation proves to be insufficient, this deficiency being covered at the expense of a flow off from the leaves of previously stored plastic substances. In the plants, exposed during this period to soil drought, the increment of dry substance in the upper internodes of the stalk, and in the spikes, does not counterbalance the loss of dry substance by the leaves and lower internodes of the stalk.

All the data quoted above show that in plants exposed to drought during the period of embryonal development of the spike and flower organs, spikes with a reduced number of spikelets are formed. During the period of internal differentiation and formation of flower organs, when the embryos of the subsequent spikelets had already formed, the influence of drought is such that a number of spikelets either remain undeveloped, or attain nearly a normal size, remaining sterile. About the same picture is observed under the influence of drought during the period of the most intense growth and ripening of the generative organs, with the only difference that in this case the sex organs of the flowers are mostly affected, and more than half of the spikelets in the ear remain sterile. In this series of plants the maximum decrease of grain yield is observed.

Table 3

Action of Drought on Redistribution of Plastic Substances Within the Plant During the Period of the Most Intense Growth of Spike and Stalk

Series of plants	Day of harvesting after cutting-off water supply, series III	Weight of 30 stalks in g. Blades, sheaths and internodes are numbered in the order from above to the base of the stalk							
		Leaf blades		Leaf sheaths		Internodes of stalk		Spikes	Total mass
		1st and 2nd	3d and 4th	1st and 2nd	3d and 4th	1st and 2nd	3d and 4th		
Control	4th	3.55	2.91	4.66	2.00	2.13	6.15	4.05	25.45
	7th	3.39	2.29	3.39	1.87	4.49	6.08	4.93	26.44
	10th	3.40	2.55	3.40	1.84	5.71	6.21	5.61	28.72
	Decrease (—) or increase (+) of weight during 6 days	−0.15	−0.36	−1.26	−0.16	+3.58	+0.06	+1.56	+3.27
III	4th	3.60	2.31	4.62	1.80	1.96	5.65	4.15	24.09
	7th	3.26	2.06	4.32	1.82	3.41	4.41	5.06	24.34
	10th	2.89	1.87	4.00	1.55	3.34	4.81	5.32	23.78
	Decrease (—) or increase (+) of weight during 6 days	−0.71	−0.44	−0.62	−0.25	+1.38	−0.84	+1.17	−0.31

From the above said and data available in literature ([1], [2], [6]) it may be considered established that plants of spring wheats become particularly sensitive to insufficient water supply at the period of the most intense growth of generative organs and stalk (last third of the phase of stalk formation until flowering). From Table 3 the physiological cause of this phenomena is obvious. The stalk and spike with all its parts, growing intensely at that period, demand a comparatively larger amount of plastic substances; but under the conditions of soil drought they are nourished chiefly at the expense of products previously stored in leaves and in the lower internodes of the stalks; therefore, they naturally suffer a deficiency in plastic material. Owing to this, the normal course of the development of the sexual organs of the flower is broken and number of them become incapable of fertilization. The biochemical nature of these phenomena will be revealed by further experiments.

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PHYTOPATOLOGY

THE EFFECT OF *PHYTOPHTHORA INFESTANS* DE BARY ON THE FERMENTS OF AFFECTED PLANTS

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Phytophthora infestans is a specialized parasitic fungus which causes dry rot of several species of *Solanum*. The cases of epidemic of *phytophthora* rot affecting the cultivated forms of potato have been recorded since the forties of the last century (De Bary, 1862, Yatchevsky, 1933, Salaman, 1937).

The morphological descriptions and biological features of *Phytophthora infestans* (=*Peronospora infestans*) together with the anatomical investigation of the affected potatoes are given in the works of A. de Bary (1881), which have not lost their significance up to now.

The examination of *Phytophthora infestans* under saprophytic conditions was initiated by Hecke (1898) and Matruchot and Molliard (1900). Hecke obtained the pure cultures of *Phytophthora infestans* on different broths; their growth was delayed on mediums containing a great amount of soluble substances; osmotic pressure was not determined. Matruchot and Molliard (1908) being unaware of Hecke's work conducted likewise the investigation of the development of *Phytophthora infestans* on different broths and on the sterilized slices of various boiled vegetables (pumpkin, melon, potato, etc.). The growth did not take place on the boiled potato. Brefeld (1883) prior to the publication of the works of Hecke and Matruchot and Molliard considered *Phytophthora infestans* a semi-parasitic fungus and afterwards received (1908) a vigorous growth of the fungus in the extract from the dried potato tubers obtained by means of cold extraction with the addition of beer must. Clinton (1911) and Pethybridge and Murphy (1913) observed the formation of oöspores of *Phytophthora infestans* grown on various agarized mediums; the question, however, remained unsettled whether the oöspores were formed in result of a sexual process or parthenogenetically.

The biochemical features of *Phytophthora infestans* were not investigated. The chemical examination of the potato tubers affected by *Phytophthora infestans* was conducted by Lepik (1929); he observed in the diseased tubers the decrease of starch and the increase of sugar content. Vigorous growth of *Phytophthora infestans* on the potato tubers, accompanied by the formation of sugar, suggested the presence of active amylase in the fungus. In our experiments, however, the sowings on the mediums containing starch—as the source of carbon—always gave negative results, the fungus being undeveloped. The growth of *Phytophthora infestans* proceeded after the sterile introduction of diastase into the medium had been performed. Immediate test for

amylase in the mycelium showed its absence. It was found to be of interest to reveal the mechanism of the effects of *Phytophthora infestans* on potato plants and to give it a physiological interpretation, especially with regard to the mobilization of reserve starch in the affected plants.

Haehn and Schweigart's investigations of amylase in the potato tubers (1923) showed its low activity which becomes still lower at dialysis. Minagawa obtained (1933, 1934) a preparation of the ferment of amylosyntease from rice and potato tubers. Amylosyntease converted, according to Minagawa, dextrins into starch, showed a low temperature optimum and was completely destroyed at 50°C. Our determinations of amylase in the tubers and foliage of the potato are in complete agreement with the data obtained by Haehn and Schweigart.

The low activity of the potato amylase may be ascribed either to the low content of the ferment or to the presence of inhibitors of reaction. To the latter belong the inhibitors of many fermentative reactions, non-specific for amylase only, as for instance tannins (Oparin and others, 1928, 1933, 1934), and the specific inhibitors of amylolytic reaction. Among these inhibitors may be classed as well Minagawa's amylosyntease and sistoamylase discovered by Chrzaszcz and Janicki (1933) in seeds and shoots of buckwheat and cereals. Sistoamylase is an inhibitor of amylolytic reaction at all its stages: the hydration of starch, dextrination and saccharification. To elucidate the question, whether the ihnibitors of amylase are present in potato tubers, several qualitative and quantitative experiments were conducted by the authors. We proceed now to describe these experiments and to state the results obtained.

The tubers of four varieties of potato with different degrees of resistance to *Phytophthora infestans* were taken: Early Rose (extremely low resistance), Lorch (low resistance), Voltman (average resistance) and 8670 JKH (resistant). Water extracts (1:2) were obtained from these tubers after their having been carefully ground.

4 cm^3 of extract were introduced into each of the flasks filled with 25 cm^3 of 1% solution of soluble starch +1 cm of $n/10$ phosphate buffer solution (with pH=7.0)+3 drops of 0.5% malt diastase solution.

All solutions had been previously heated to 45°C, after which the mixtures were placed in a water bath with the temperature of 45°C. The rate of dextrination was determined by iodine method (Wohlgemut).

The results are as follows:

Table 1

	Control (diastase + + water)	Extract from Early Rose + + diastase	Extract from Lorch + diastase	Extract from Voltman+diastase	Extract from 8670 + diastase
0 minutes	deep blue violet	deep blue violet	deep blue violet	deep blue violet	deep blue violet
10 minutes	red	violet	violet-red	deep blue violet	deep blue violet
20 minutes	yellow	violet	red	deep blue violet	deep blue violet

All the extracts slowed up the rate of dextrination of starch; especially those obtained from the tubers of Voltman variety and from 8670 JKH. Hence we may conclude that potato tubers contain the inhibitors of amylase. The action of the inhibitor was not stopped by adding the solutions of gela-

tine and peptone to the extracts, i. e. it cannot be considered as a tannine, whereas the boiling of the extracts lowered its effect considerably. Most likely this inhibitor has to be classed among specific inhibitors. Further on we shall name it sistoamylase.

For determination of the power of sistoamylase to inhibit the process of saccharification we have made a series of quantitative determinations by the following method: into 3 round-bottomed flasks with 150 cm³ capacity were brought:

Flask I — 15 cm³ of extract from potato plant (1 : 2)
+ 30 cm³ of 2% starch solution
+ 4.5 cm³ *n*/10 phosphate buffer solution with pH=7.0
+0.5 cm³ of distilled water

Flask II — 15 cm³ of extract
+ 30 cm³ of 2% starch solution
+ 4.5 cm³ of phosphate buffer solution, pH=7.0
+ 0.5 cm³ of 1% diastase dissolved in 2% NaCl solution

Flask III — 15 cm³ of distilled water
+ 30 cm³ of 2% starch solution
+ 4.5 cm³ of phosphate buffer solution, pH=7.0
+ 0.5 cm³ of 1% diastase solution

All solutions previously heated to 45°C were mixed in the ratios mentioned above and kept in a water thermostat, at 45°C, during 30 minutes.

For 10 cm³ of mixture an iodometric determination of sugar (=glucose+ +maltose) content can be made according to Wilshtetter. The hydrolysis proceeds in the flask III in the presence of a diastase preparation alone; in the flask I, under the action of extract diastase; and in the flask II, under the action of a diastase preparation+extract diastase.

Summing up the sugar contents (in 10 cm³) of flasks III and I and subtracting from the whole the sugars contained in 10 cm³ of the mixture from the flask II, we obtain the relative quantity of sistoamylase in terms of cm³ of *n*/10 I. We may illustrate this calculation by the following scheme: III—sugars in 10 cm³+I—sugars in 10 cm³—II—sugars in 10 cm³=the relative quantity of sistoamylase in 1 cm³ of the tissue sap.

We proceed now to state the quantitative determinations of sistoamylase.

We took tubers of five potato varieties with various degrees of resistance to *Phytophthora infestans*. The tubers were received in spring from the storehouse of the Scientific Potato Institute (near Moscow) without any injuries and noticeable signs of sprouting. The results of the determinations of amylase and sistoamylase are presented in Table 2.

It follows from the figures mentioned above that the tubers of all potato varieties contain sistoamylase, inhibiting the reaction of starch saccharification by amylase; in different varieties the quantity of sistoamylase is not always identical, being high in resistant varieties and relatively low in non resistant ones. The activity of amylase in the tubers of all potato varieties is rather low.

The quantity of sistoamylase is not constant, but undergoes noticeable changes in the tubers, at rest.

Some of the tubers of one potato plant grown in an earthen vessel with a small amount of soil were formed on the surface of soil; they turned green under the influence of light and began to sprout on the maternal plant. The tubers covered by soil did not sprout at all and had a normal appearance. To determine sistoamylase the tubers were divided into three groups—sprouting, poorly sprouting and unsprouting tubers.

The results of the determinations are presented in table 3.

Table 2

Preparations of amylase and sistoamylase	Quantity of sugars in cm^3 of $n/10$ I	Amount of sistoamylase in cm^3 of $n/10$ I	Resistance of potato-varieties to *Phytophthora infestans*
Diastase alone	9.00	—	
The extract from Early Rose	0.35	—	Extremely low resistant
The extract from Early Rose + diastase	7.60	1.75	
The extract from Epicurus	0.50	—	Low resistant
The extract from Epicurus + diastase	7.50	2.00	
The extract from Lorch	0.45	—	Low resistant (more resistant than Epicurus)
The extract from Lorch + diastase	7.40	2.05	
The extract from Smislovsky	0.65	—	With regard to resistance approaches to Lorch
The extract from Smislovsky + diastase	7.00	2.65	
The extract from Voltman	0.60	—	Fairly good resistance (more resistant than the preceding variety)
The extract from Voltman + diastase	6.40	3.20	

Table 3

Preparations of amylase and sistoamylase	Quantity of sugars in cm^3 of $n/10$ I	Quantity of sistoamylase in cm^3 of $n/10$ I
Diastase alone	8.90	—
Extract from sprouting tubers	1.90	—
Extract from sprouting tubers + diastase	8.40	1.10
Extract from poorly sprouting tubers	1.30	—
Extract from poorly sprouting tubers + diastase	7.10	3.10
Extract from unsprouting tubers	0.80	—
Extract from unsprouting tubers + diastase	6.70	3.00

With the sprouting of tubers the activity of amylase rises while that of sistoamylase diminishes. It is well known fact that the sprouting of tubers is accompanied by the accumulation of sugars, especially in growth buds.

As we may observe, sistoamylase plays a certain part in the processes of converting the carbohydrates of tubers. Sistoamylase was discovered by the authors in the leaves of the potato plants, gathered in the period of maximum tuber-formation (1. VIII).

Below we give the indexes of sistoamylase for the leaves of three varieties of potato different as to their resistance to *Phytophthora infestans*: *Solanum demissum*—an immune variety, 8670 JKH—a highly resistant variety and Early Rose—a variety with extremely low resistance.

The results of these determinations are summarized in table 4.

Table 4

Preparations of amylase and sistoamylase	Quantity of sugars in cm^3 of $n/10$ I	Quantity of sistoamylase in cm^3 of $n/10$ I
Diastase alone	7.30	—
Extract from *Sol. demissum*	2.60	—
Extract from *Sol. demissum* + diastase	2.80	7.10
Extract from 8670 JKH	1.90	—
Extract from 8670 JKH + diastase	2.40	6.80
Extract from Early Rose	1.30	—
Extract from Early Rose + diastase	3.40	5.20

Let us now examine the results of the sistoamylase determinations in the tubers infected by *Phytophthora infestans*. The experiment was conducted with Lorch variety. The tubers, after their surface had been rubbed with alcohol, were cut with a sterile lancet into rounds 0.5 cm thick, which were infected at different terms by introducing oöspores in water drops.

By the time of analysis the growth of the fungus was different, from the very first stages (the beginning of darkening of the potato surface) up to the full overgrowth of the infected rounds by mycelium.

The quantity of amylase and sistoamylase in the infected tubers is given in Table 5.

Table 5

Preparations of amylase and sistoamylase	Quantity of sugars in cm^3 of $n/10$ I	Quantity of sistoamylase in cm^3 of $n/10$ I
Diastase only	8.20	—
Extract from a sound tuber	0.30	—
Extract from a sound tuber + diastase	5.00	3.50 (100.0)
Extract from an infected tuber (the beginning of darkening, infection period 24 hours)	0.30	—
Extract from an infected tuber (infection period 24 hours) + diastase	5.30	3.20 (91.4)
Extract from an infected tuber (mycelium on borders, infection period 3 days)	0.30	—
Extract from an infected tuber (infection period 3 days) + diastase	5.60	2.90 (82.8)
Extract from an infected tuber (overgrown with mycelium, infection period 5 days)	0.50	—
Extract from an infected tuber (infection period 5 days) + diastase	7.10	1.60 (45.7)

The development of *Phytophthora infestans* on tubers is accompanied by gradual decrease of sistoamylase. This peculiarity of *Phytophthora* seems to play a prominent part in its biology and contributes considerably to its destroying influence on potato growing.

Phytophthora infestans, diminishing the amount of sistoamylase, excludes the possibility of rapid rising of osmotic pressure in the medium; the presence of sistoamylase hinders the development of other microbes. The

growth of *Phythophthora infestans* on tubers lasts but a short period, the considerable decrease of sistoamylase admitting infestion of the tubers by putrefactive microbes, and thus, *Phythophthora infestans* initiates the process of putrefaction (Appel, 1905—1908).

Sistoamylase was discovered in the tubers and leaves of cultivated and wild forms of potato.

The decrease of sistoamylase and low activity of amylase are observed at the infection of potatoes by *Phytophthora infestans*.

The dependence of the resistance of potato varieties to *Phytophthora infestans* and on sistoamylase was noted.

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PLANT PHYSIOLOGY

INTERMEDIATE PERIOD OF DEVELOPMENT AND ANATOMIC-MORPHOLOGICAL CHARACTERS OF LIGHT STAGE IN WHEAT

By V. T. EREMENKO

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There still remain many obscure phenomena we meet with in turning to account the stage development for cultivation of farm crops. Such phenomena may refer to the behaviour of plants in different environment, as well as to the revelation of such characters and properties of plants which are typical for a given developmental stage, and may serve as bearings in experimental and practical work.

To clear some of these questions a series of experiments was conducted in the Department of Agrophysiology of the Ukrainian Institute of Socialist Agriculture. The data obtained are exposed in the present article.

The work that opened this series was that of Matskov, Shimanski and Trigubenko (1). From data obtained in 1933, the authors assumed that in the summer wheat *Hordeiforme* 010 an intermediate stage between yarovization and light stage may exist.

The idea of the existence of an intermediate period of development was anew suggested in 1934 in connection with the study of the length of yarovization and vegetative periods under the action of stimulators(2).

In order to check this assumption a special experiment was carried out in 1935 with the winter wheat Novokrymka 0204 (*Trit. vulg.* var. *erytrospermum*) (3). This experiment also proved that during the first 12—15 days after germination the length of the vegetative period did not depend upon day length. However, the first 10—13 days of the experiments happened to be unfavourable with regard to temperature necessary to the second stage of development—that of light. It was, therefore, impossible to do anything but speculate on the existence of an intermediate period of development.

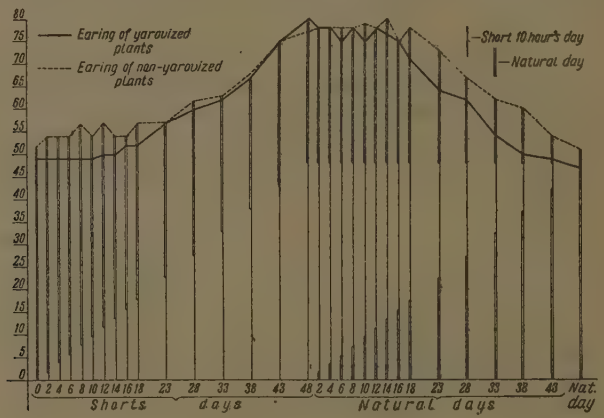
In this connection in 1936 an experiment was set up under temperature conditions favourable for passing the light stage which, according to T. D. Lysenko (4), can proceed at a high temperature (above 5—10°C), and under long day conditions only. In a paper by Rasumov(5) we also find a statement that at temperatures below 7—9°C the light stage does proceed, yet too slowly.

The experiment was conducted with spring wheat *Lutescens* 062 (early ripening), and *Hordeiforme* 010 (late ripening), and the winter wheat Novokrymka 0204. Sowing was done on April 24 with yarovized seeds. The plants were grown in vessels. During the first 25 days after sprouting the minimum temperature was about 17°C, and the maximum from 23 to 35°C. However, in this experiment too, the plants did not respond to the length of

the day direct after the appearance of sprouts, but several days after (in *Lutescens* 062 and Novokrymka 0204 after 8 to 10 days, and in *Hordeiforme* 010 after 15 to 18 days).

There are no reasons to admit that the plants passed the stage of yarovization since in the experiment exclusively seeds sure to have been previously yarovized were used. Thus, the experimental data obtained in 1936, did also support the assumption of an intermediate period in wheat development. In 1937, experiments were conducted with the same purpose, viz. to find out the existence of an intermediate period of development, and to reveal the initial and final anatomic-morphological characters of the second, light, stage.

The experiment was carried out both under field conditions and in vessels. Seeds sown in field belonged to the summer variety *Hordeiforme* 0802. Part of them had not passed yarovization, but had weakly germinated before sowing, others had been yarovized at 4—5°C during 23 days. Sowing was done with great delay, to eliminate the influence of low temperatures. During the first days after sprouting the minimum temperature ranged from 7 to 15°C,



and the maximum—from 21—32°C. Sprouts appeared on May 13. Since the day of intense sprouting the plants of the first 16 rows of the wheat were given short 10-hour days. This was achieved by covering the entire plot with a large dark ply-wood chamber from 5 p. m. until 7 a. m. of the following day.

Owing to successive rearrangements of the chamber (see scheme), on one part of the plot the plants started growing being for different number of days given a short day, and next a natural day. On the other part of the plot the plants of different rows were at first given a various number of natural days subsequently, and afterwards were shifted to a short 10 hour day. 48 days after sprouting, the plants of all variants were transferred to natural day. Until the 30th day after sprouting there was hardly any exterior difference among plants belonging to different variants, but further on the differences became more and more pronounced. On the scheme the curve of earing of the experimental plants is represented according to variants. It may be seen from this scheme that the plants from yarovized seeds given 2 to 14 short days after sprouting eared almost simultaneously with the control plants—on the 49th and 50th day. Plants which were given a short day during more than 14 days, proceeded to earing the later, the longer they were given short day. Experimental plants, given a maximum number of short days

Influence of Light Regime on the Rate of Earing in Wheat
Hordeiforme 0802

Pot experiment, 1937

	From sprouting	Short 10 hour days											
		0	2	4	6	8	10	12	14	16	18	20	Short day throughout the experiment
Number of days from sprouting to earing	Yarovized	50	51	51	51	51	51	52	53	53	54	54	126
	Unyarovized	54	55	54	54	54	54	55	56	56	56	56	130

	From sprouting	Natural days															
		2	4	6	8	10	12	14	16	18	20	25	30	35	40	45	Natural day throughout the experiment
Number of days from sprouting to earing	Yarovized	119	128	121	133	127	120	114	111	112	108	104	74	59	53	53	50
	Unyarovized	—	135	131	135	141	138	135	135	139	135	119	91	68	63	59	54

(48), formed ears on the 80th day only, the earing in individual specimens being not unanimous, retarded. Observations of the condition of the point of growth in the control, yarovized plants showed it to change scarcely during the first 15—18 days; only beginning with the 18th day a differentiation of the rudimentary spike was noticed. In unyarovized plants the differentiation in the cone of growth was noticed several days later. Thus was made clear the connection between the initial stages in the development of the spike, and the beginning of the plants response to the day length manifested in changing rates of development. In order to check this phenomenon a supplementary experiment with the same wheat *Hordeiforme* 0802 was performed in 1937. This experiment has also evidenced the connection between the outset of spike differentiation, and the first response of the plant to the day length. The time agreement between the beginning of the rudimentary ear differentiation and the strong change in the plant's response to the day length is far from being random. This moment in the plant development should be regarded as a transitional biological stage during which the plant is acquiring a new quality, and which seems to be the outset of the light stage.

As to the scheme, it is necessary to notice that the yarovized plants grown the first 2—14 days after sprouting under natural day conditions, and further on, until the 48th day, under short day conditions, eared at the same time with the plants which were given short day all the 48 days through, viz., during the first 14 days the plants did not react to the long day. The plants of the subsequent variants, which had first been given 16—28 natural days, began to form ears the earlier, the later they were shifted to a short day.

Finally, the plants which were given short day since the 38th day after sprouting, when they already had a well expressed pipe, eared at dates approaching those of the earing of controls.

In unyarovized plants, the general course of development according to the variants coincided with the development of the yarovized ones. They eared noticeably (about 5 days) later, especially the variants with a short vegetative period.

The pot experiment with the same sort of summer wheat *Hordeiforme* 0802 was conducted according to a scheme closely approaching that of the field experiment. In the table is shown the time of earing of the bulk of experimental plants, according to variants.

It is seen from this table that during the first 12 days after sprouting the plants did not react at all, while from the 12th to the 20th day they reacted but weakly to the day length by a change in the time of earing. The change of the day length from the 20th to the 35th day after sprouting had a strong influence on the time of earing, and again, a weak influence beginning from the 35th day after sprouting, i. e., when the plants had a well expressed pipe. The yarovized plants in the pot experiment lagged behind the yarovized ones in development, and reacted somewhat later to the change in day length. Thus, for each variant the behaviour of the plants in the pot experiment closely approached the behaviour of the plants in the field experiment.

Conclusions

1. From the experiments carried out in 1935—1937 by the Department of Agrophysiology of the Ukrainian Institute of Socialist Agriculture, it should be assumed that in wheat there exists a biologically important transitional stage of development between the yarovization and light stages.

At this period wheat plants do not need any long days, which are one of the most essential factors required at the following developmental stage, viz. the light stage.

A typical anatomic character of the transitional period is the lack of differentiation of the rudimentary spike.

2. Having passed the transitional period of development, the plants enter the second, light, stage, which alongside of other exterior factors requires either continuous illumination or long day. Plants given a short day at that period, stay behind in earing.

According to our data, the outset of light stage is characterized both by increased long day requirement, and by the outset of rudimentary spike differentiation. The end of this stage corresponds to the beginning of pipe-formation when the differentiation of the main elements of the ear seems to be achieved.

After the end of the light stage, the requirement of the plant in long days seems to drop.

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PLANT PHYSIOLOGY

MOTION OF BLOSSOM HORMONE IN GIRDLED AND GRAFTED PLANTS

By M. Ch. ČAJLACHJAN

(Communicated by A. A. Richter, Member of the Academy, 17. II. 1938)

The results of experiments carried on during 1932—1936 and summarized in author's «Hormonal Theory of Plant Development» (1) show that sexual maturity, blossoming and fruiting are connected with the accumulation of a sexual or blossom hormone for which the name «florigen» has been suggested.

All of the principal statements advanced in (1) are briefly exposed in our last paper (2) printed in C. R. Acad. Sci. URSS, while some of them are to be found in the paper of Garner (3). That part of the work which has been concerned with the case of blossom hormone passing from one plant to another after transplantation is supported by experiments made by J. Kuijper and L. K. Wiersum, B. Moshkov and G. Melchers. J. Kuijper and L. K. Wiersum (9) worked with «short day» plants of *Glycine Max*. L., they grafted blossoming scions subjected to a short day regime upon long day stocks, which began then to blossom. The inverse grafting of long (normal) day scions upon blossoming short day stocks made the scions blossom.

Moshkov (6, 7) obtained flower-buds on the stock in grafting the tops of blossoming plants of Samsun tobacco upon decapitated, not blossoming Maryland Mammoth; Melchers (10) provoked blossoming in perennial plants of *Hyoscyamus niger* by grafting on them the blossoming scions of the same plant or of *Hyoscyamus albus*. Blossoming grafts of *Petunia* and of *Nicotiana tabacum* called forth the same effect.

Comparing his own experiments on grafting with the results of my tests, Moshkov subjects the latter to criticism. An answer to this criticism is given in my article (5), which also deals with the questions raised by Melchers in discussing my experiments with grafts. In 1937 I prosecuted the study of the plant development processes as influenced by hormones and was successfull in clearing up the question as to the motion of the hormone «florigen» in girdled and grafted plants. The experiments were carried on with *Perilla nankinensis*. Some of the results are reported below.

1. Motion of the blossom hormone in girdled plants. Plants of Perilla in earth vessels, sown on 17. III. 1937 were used. Following the appearance of young growth the plants received during night electric light, supplementary to the day light until 25. IV, and then the natural day light only. Owing to this the plants remained all the time in the phase of vegetative growth, and by the beginning of July they were 70 to 80 cm high. On the 2. and 3. VII the preparation of the plants to experiments was done.

All the plants were decapitated and divided into two groups: in the plants

of the I group the upper leaves were removed, the lower 6—8 leaves being left; all the shoots were removed, except the two uppermost located one opposite the other. In the plants of the II group, on the contrary, the lower leaves were removed and the upper 6—8 leaves were left; all the shoots were removed, except the two lower, somewhat above ground level.

Thus, in the I group the leaves in the plants are located below, and the shoots «indicators», above; in the II group the leaves are located above, and the shoots «indicators», below. Such a position was maintained throughout the experiment by pruning the shoots appearing anew every 5 days. In the I group of plants the ascent of «florigen», in the II group its descent was studied.

In each of the groups 3 kinds of experimental plants were twice subjected to test: 1) Leaves of a plant having a short, 10-hour day, and shoots having a full (long) day. 2) Same, as 1, but between the leaves and shoots an incision in the stem was made, somewhat deeper than the half diameter of the stem and 4 to 5 mm wide. The incision was made on that side of the stem where one of the shoots was located (the right one in relation to the number on the vessel). 3) Same, as 1, but between the leaves and the shoots a circular incision was made in the bark, 4 to 5 mm wide. In girdling the bark was removed from the tetrahedral stem of Perilla with a pruning knife, and the upper layer of wood was slightly scrapped off, lest any bark elements should remain. Full (long) day plants served as control.

The tested plants remained under the influence of a full day until 27. VIII, afterward up to the end of the experiments they received supplementary electric light, which was switched in with the sunset, and switched off at midnight. When the plants were placed under the conditions of a short 10-hour day, they were covered up from 6 p. m. until 8 a. m. with cloth hoods made of double, thick stuff, white on the out side and black inside. The chief purpose of the experiments was to call forth in the leaves influenced by the short day the formation of the blossom hormone, and to follow its motion towards the shoots—«indicators»—in an uninjured stem ([1]), in a stem incised on one-side ([2]), and in a girdled one ([3]).

The operation of plants was performed on July 4; the same day the covering with the hoods was begun. Owing to lopping off the top and shoots, the two shoots left on the plants rapidly tarted growing; new large leaves appeared on the shoots. To make apparent the reaction of the shoots all the new large leaves appearing on them were lopped off.

No difference in the behaviour of the shoots in the experimental and control plants was observed until July 31. At this day, however, in the I group of plants flower-buds appeared on the shoots with short day leaves ([1]) and blossoming set in on July 12—13, while in the II group the flower-buds appeared on the respective shoots ([1]) on July 3 and broke out into blossom on August 15—18. The shoots of the control day plants gave neither flower-buds, nor blossoms until the end of the experiments. Data concerning the development of shoots in the control and experimental plants are shown in the table. The state of the shoots at the end of the experiments is shown on Fig. 1 and 2.

The leaves of all the plants shown in Fig. 1 were under a short day. On the left the plant has a whole stem—both shoots are blossoming; in the middle, a laterally incised stem—both shoots are blossoming too; on the right side a girdled stem shows vegetating shoots. Fig. 2 offers the same picture in the plants with the shoots under the leaves. The control plants, not shown on the photos, have vegetating shoots in both cases.

As may be seen from the photographs and table, girdling impedes both the descent and ascent of «florigen» in the stem—the shoots do not blossom. They do, however, in plants with laterally incised stem; in this case the shoot

Development of Perilla Shoots in Connection with Girdling and Lateral Incision into the Stem

№ of variants	Experiment	Beginning of experiment	Formation of flower-buds		Blossoming	
			Left shoot	Right shoot	Left shoot	Right shoot
	I Group. Leaves below, under 10-hour day; shoots above, under full day					
K	Full day, control	4.VII	—	—	—	—
1	Stem uninjured	4.VII	31.VII	31.VII	13.VIII	12.VIII
2	Stem incised laterally below the right shoot	4.VII	31.VII	31.VII	12.VIII	13.VIII
3	Girdling	4.VII	—	—	—	—
	II Group. Leaves above, under 10-hour day; shoots below, under full day					
K	Full day, control	4.VII	—	—	—	—
1	Stem uninjured	4.VII	3.VII	3.VIII	15.VIII	18.VIII
2	Stem incised laterally above the right shoot	4.VII	8.VIII	8.VIII	16.VIII	19.VIII
3	Girdling	4.VII	—	—	—	—

on the side of incision blossoms at the expense of the hormone supplied through the bark of the opposite side, i. e. moving not only lengthwise, but in the lateral direction too. In experiments of 1936 with *Chrysanthemum*

Fig. 1. Development of upper shoots on a girdled or laterally incised stem (Photo taken on September 1, 1937).

indicum we have established that in basipetal direction «florigen» descends through the bark (1).

The present experiments show that the blossom hormone both ascends and descends the stem in the bark; that it moves in the bark not only lengthwise by the chief ducts, but also in lateral direction.

2. Motion of the Blossom Hormone in Grafted Plants. For experiments with grafts Perilla plants of the same date of sowing (March 2) were taken as in the previous experiments. The plant was kept in a hot-house until May 19 being exposed to supplementary electric light, up to April 25.

On May 19 they were transferred to a greenhouse and divided into 2 groups: the plants of one group were left under full natural day, while the plants of the other group received a short 10-hour day, being every day at 6 p. m.

rolled in on trucks into a dark hut, and at 8 a. m. exposed again to day light. Under such different conditions of light the plants remained until June 15; the short day plants began to form flower-buds on June 12, and broke out into blossom on June 20; the long (natural) day plants remained in a vegetative state.

On June 15 tops and cuttings of vegetating plants were grafted (cleft grafting) in the plants beginning to blossom, and vice versa. After grafting, all the experiment and control plants remained under conditions of natural (long) day; following August 27 they were given electric light too.

During the time from May 19 till June 15 the blossom hormone «florigen» accumulated in the leaves of the short day plants; part of it was used up in flower-bud and flower formation. In the leaves of full day plants there was no «florigen», and the growing points developed vegetatively. Thus, in the I group of grafts the short day stocks served as a source of stored up «florigen», while the full day scions were the indicators of «florigen» action; in the II group

Fig. 2. Development of lower shoots on a girdled or laterally incised stem (Photo taken on August 23, 1937).

of reverse grafts the short day scions served as a source of «florigen»; and the full day stocks were the indicators. The chief purpose of the experiments was to find out the part played by leaves and shoots in both kinds of grafting.

As might be expected from the experiments of 1936 ([3], [4]) the blossoming component exerted a marked influence upon the development of the vegetative component. Already on July 31, the long day tops grafted into blossoming stocks began to form flower-buds; on September 11 they blossomed, and by the end of that month produced ripe seeds. In the control grafting of long day tops into vegetating long day stocks there was no formation of flower-buds or blossoms up to the end of the experiments. Similar results were obtained in the reverse grafting; here, under the influence of the blossoming tops, the long (full) day stocks began to form flower-buds on July 31, they blossomed on September 11, and by the end of September yielded ripe seeds. The control grafts did not show any formation of flower-buds or blossoms up to the end of the experiment.

Yet, not in all the cases did clear and rapid reaction set in. The results of the experiments show that the presence of leaves and shoots on the grafts was determinant in such cases. The regularity here observed was found valid both for the direct and reverse grafting.

The travelling of the principal mass of «florigen» via the point of grafting occurs in the case when leaves are left on the blossoming component, and shoots with growing points on the vegetating one. If shoots are left on the blossoming component, a small amount of «florigen» passes to the vegetating component because its main mass is used up by the shoots of the blossoming component. If the shoots are removed on the blossoming component, the leaves being left on the vegetating one, the greater part of the «florigen» passes to the latter; but its concentration is considerably decreased owing to a large amount of plastic structural substances produced by the leaves of the vegetating component. The effect of «florigen» action is, therefore, reduced. In the case, when shoots and leaves are left on both the components, they behave to a considerable extent as independent organisms: the flowering component blossoms and bears fruit while the vegetating component, in the majority of cases, vegetates.

Besides grafting and splicing of pieces of plants, experiments were started, in which only the leaves of flowering plants were transplanted onto vegetating long day plants. Preliminary experiments carried out by us in 1936 have shown that growing-in occurs best of all when leaves are grafted upon the stem on a scion. Experiments in 1937 were started thus: the top and large leaves of the vegetating long day plant were removed and a scion with two opposite «flower-ripe» leaves, taken from a blossoming plant, was inserted into the cleft. The buds located in the axils of these leaves were removed, prior to grafting. During the one day of June 15 a total of eight such grafts were made. The vegetating long day plants served as a control, being grafted beforhand (on May 20) with leaves also from vegetating plants; control graftings have been performed.

Fig. 3. Blossoming and fruiting of plants under the influence of grafted «flower-ripe» leaves (September 25, 1937).

On the control plants the grafted leaves grew-in very well, the shoots on the plants started growing, and up to the very end of the experiments they formed neither flower-buds, nor blossoms. In the test plants, the grafted leaves also grew-in readily, and the shoots started growing, but very soon under the influence of the «flower-ripe» leaves, the vegetative growth of the shoots ceased, and they began to form flower-buds and to blossom. Bud formation of the shoots began on all the plants on July 17—August 4; blossoming of some plants began on August 19, and of others on September 1 and 11. We are able to judge about the state of the test plants on September 25 from Fig. 3; the latter respresents one long day plant with shoots blossoming under the influence of the «flower ripe» leaves grafted on it; these leaves were cut off together with the scion from a blossoming plant. One grafted leaf is left on the plant and the other is cut off.

In the course of time, the grafted leaves wilted, dryed up and fell off. In the control grafts wilting and drying up of the leaves began on August 20, i. e. 92 day later (20. V—20. VIII) after they had been grafted to the plants. Formation of flower-buds on the test plants began on July 25, i. e. 40 days (15. VI—25. VII) after the «flower-ripe» leaves had been grafted to them.

The grafted leaves on the control plants fell off earlier (beginning of September) than in the test ones (end of September), but the total amount of time during which they remained on the plant was equal for both.

In Fig. 4 is shown the state of the plants on October 19. On the left, is a control plant with vegetative shoots, on the right—a test plant with ripe fruit. In both plants the grafted leaves have already fallen off. Such a sharp metamorphosis of the plant—its transformation out of a vegetating plant into a fruiting one—occurred at the expense of that amount of the blossom hormone, which had time to accumulate in a pair of leaves during 27 days

Fig. 4. Fruiting of plants under the influence of grafted «flower-ripe» leaves. Experimental plant on the right, control on the left. In both plants the grafted leaves have already fallen off (October 19, 1937).

(19. V—15. VI). This occurred under the influence of a short day before these leaves had been cut off and grafted upon the vegetating plant.

Thus, under favourable conditions of light such an amount of blossom hormone is accumulated in the leaves as make pair of such «flower-ripe» leaves sufficient to call forth the metamorphosis of a whole plant —its blossoming and fruiting. Girdling and grafting of plants are the fundamental methods in the practice of horticulture. These are the means to achieve an acceleration of blossoming and fruiting in perennial fruit plants. It may be assumed that the regularity of the processes connected with girdling and grafting is also valid in the physiology of perennial plants.

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PLANT ANATOMY

FULL AND MEAGRE GRAIN IN SOFT WHEAT

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Full and meagre grain are not to be distinguished prima facie both in hard and soft wheat, a thorough analysis of various symptoms of meagreness being only able to discover differences in rather essential details of their structure. This is especially true of comparative anatomic analysis.

In the present paper is stated part of the results of the anatomic study of various meagre grains taken from the standard sort *Lutescens* 062, supplied by the Department of Geographical Cultures of the All-Union Plant Breeding Institute. Among soft wheats the sort *Lutescens* 062 is distinguished by a large degree of variability of a number of economically valuable characters of the grain; this indicates the wide range of plasticity of the sort. The structure of different meagre grains occurring in hard wheat has been described by us in our report (1), where method of research is given too.

Fragment *1* represents sections of the integument and endosperm in full-grown and well-rounded out grain. In the pericarp three layers of anatomic elements of the epi-mesocarp may be seen quite clearly, with depositions of semicellulose in their cavities. Quite distinct is the obliteration of cell cavities of these elements, and the fusion of outer layers of their integuments into continuous mass. Of all the pericarpium cells the structure seems to be best preserved in the transverse cells with closely disposed fissure-like pores.

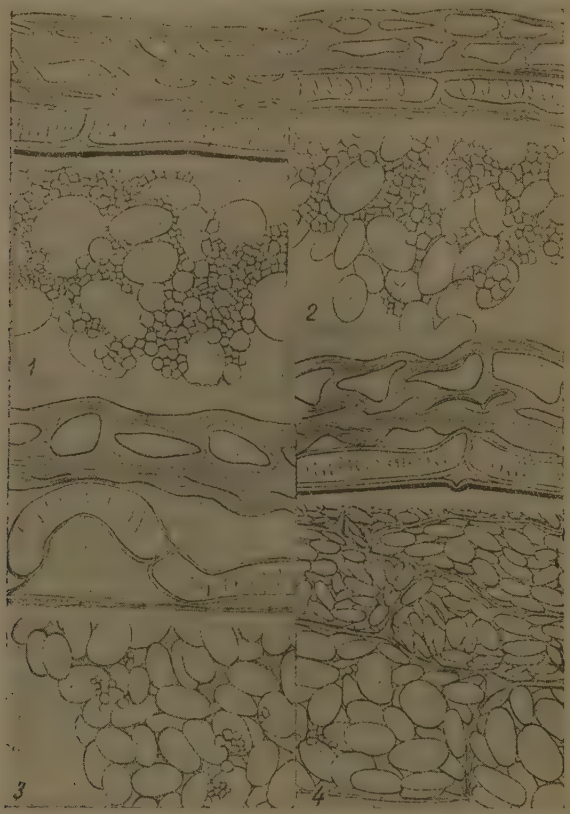
The structure of the central part of the endosperm tissue in a well-formed grain of soft wheat is, compared to hard wheat, characterized by the prevalence of polyhedral chondriosome starch, over the globular one. The polyhedral granules make up agglomerations where they are closely pressed together and are difficult to separate by an intention destruction of the endosperm tissue. Kernels with such an endosperm display characters of an intermediate structure between a flinty and a floury one, despite of their pronounced flinty character they are somewhat opalescent. Opalescence is best of all exhibited on the transverse cuts of the kernel.

Fragment *2* represents sections of the integument and endosperm of a small grain (compared with the size of the ovary), but well-rounded out, smooth and mature. There are no depositions of semicellulose in epi-mesocarp. The transverse cells are unequally developed; now and then flattened out, with side and relatively sparse pores.

In the seed coat individual cells making up the inner layer of the coat may easily be discerned. All this indicates that the processes going on in the in-

teguments and determining their ripeness did not come to an end, though the grain seems to be quite mature.

The middle part of the endosperm tissue also shows an incomplete structure, compared to the full-grown kernel, a far less amount of round chondriosome starch being stored up. As shown in (1), the round chondriosome starch stored when the small polyhedral one has already accumulated. Moreover, the accumulation of the latter in a incomplete kernel is comparatively



weak, interrupted by large interspaces filled with protein. The grain is glassy.

On the whole, both integument and the endosperm tissue of an undersized grain are characterized by some structural characters which distinguish them from a full-grown grain—characters of incomplete ripeness, so to say. Yet, these characters of incomplete ripeness of the integument and endosperm are coordinately developed and the structure of such a meagre grain is quite as harmonious, as that of a full-grown, completely mature one. A somewhat different picture is observed in shrivelled grains.

Fragment *3* represents sections of the integument and endosperm of a grown up, but strongly wrinkled grain. The structure of the endosperm is quite typical for a grain with sharply expressed symptoms of meagreness. The tissue of the endosperm is filled up nearly throughout with plastid starch grains; only sparse islets of small polyhedral chondriosome starch are met

with and still more rarely separate grains of round chondriosome starch. It is quite obvious that the maturing of the contents of the cells of endosperm tissue was suspended at one of the early stages of this process. A quite different picture presents the pericarp. In the epi-mesocarp depositions of semicellulose are seen; the pores of the transverse cells though relatively sparse, are identical as to their structure to the pores of the transverse cells of a full-formed grain. The typical curvature of the transverse cells is especially striking; this is an evidence for the cells continuing to grow in an endosperm which has already ceased growing; and not only to grow, but also differentiate and ripen. Alongside of the growing and differentiation of the transverse cells, other anatomic elements of the pericarp, i. e. of the epi-mesocarp, are of course continuing their development. In the whole, the structure of the pericarp of such a meagre grain resembles the structure of the pericarp of a full-formed and full-grown grain. Consequently, in the given case, there is no agreement between pericarp and endosperm as to their state of development, the pericarp possessing characters of a far more advanced maturity.

Still more pronounced is this disagreement in an undersized, strongly shrivelled grain. In meagre grains of such a type, two symptoms of meagreness were established: small size and shrivelling. The structure of the integument and endosperm of these grains is exemplified by fragment *4*. The cross-sections of shrivelled grains have grotesque and multiform outlines; the rounded out form of grain is out of the question. The number of cell layers in the endosperm is small.

The endosperm is almost exclusively filled with plastid starch grains; the amount of chondriosome starch is insignificant. Depositions of protein are also gluten lacking. One may only discern the remains of dried protoplasm near the starch granules. The endosperm of this grain is still farer from the state of maturing than the grain of the preceding examples (fragment *3*). The good condition of the cell integument of the endosperm tissue is conspicuous. In highly cultivated wheats, especially in sorts of soft wheat, the integument of the central part of the endosperm tissue is usually nearly dissolved in the course of the grain's maturing. The integrity of the integument is still another character indicating the immaturity of the endosperm, a very early stage of its differentiation.

The structure of the pericarpium shows that this part of the grain has left the endosperm far behind with regard to ripeness. In some anatomic elements of the epi-mesocarp depositions of semicellulose are even noticed. The depositions of semicellulose are well expressed only in full-formed, quite mature grains of highly cultivated wheats. The structure of the transverse cells is normal.

Thus, in the formation of a strongly shrivelled small grain of soft wheat, the endosperm dries at an early stage of its differentiation, whereas the fruit coat continuous for some time its development. The factors, causing the shrivelling of the grain, seem first of all to effect the endosperm, as a more tender tissue, the pericarp showing greater resistance. The shrivelling of a wheat grain, as in any cereal, accounts for the non-conformity in the development and ripening of the integuments and endosperm.

In conclusion we will point out characters which distinguish the soft wheat from the hard one. An anatomic analysis of various meagre grain specimens of hard and soft wheats has shown that the most conspicuous difference is in the reaction of the cell contents of the central part of the endosperm tissue to the action of the factors responsible for the meagreness of grain.

In full-grown grains of hard wheat, having a normal pericarp structure, the endosperm is often incompletely differentiated and shows an unsuffici-

ent accumulation of chondriosome starch, especially of the polyhedral one. This state may be called that of hidden (cryptic) meagreness.

In soft wheat, which is generally storing up noticeably more of the polyhedral chondriosome starch, compared to hard wheat, the range of the normal state of full growth is much wider. The hard wheat has a narrower range of full growth: comparatively small changes in the endosperm are sufficient to cause in the grain qualities peculiar to meagre grain. As the soft wheat is growing full, the amount of depositions of chondriosome polyhedral starch may undergo considerable changes; the grain will, however, keep the features of a normally filled one. The decrease of starch storage results in increased flintiness of the grain; on the contrary, the more abundant is the accumulation of starch, the more flowry appears the grain.

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